



**HAL**  
open science

# Observer design for nonlinear systems using LMI relaxation techniques

Shivaraj Mohite

► **To cite this version:**

Shivaraj Mohite. Observer design for nonlinear systems using LMI relaxation techniques. Automatic. Université de Lorraine, 2023. English. NNT : 2023LORR0326 . tel-04584385

**HAL Id: tel-04584385**

**<https://hal.univ-lorraine.fr/tel-04584385>**

Submitted on 23 May 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



**UNIVERSITÉ  
DE LORRAINE**

**BIBLIOTHÈQUES  
UNIVERSITAIRES**

## AVERTISSEMENT

Ce document est le fruit d'un long travail approuvé par le jury de soutenance et mis à disposition de l'ensemble de la communauté universitaire élargie.

Il est soumis à la propriété intellectuelle de l'auteur. Ceci implique une obligation de citation et de référencement lors de l'utilisation de ce document.

D'autre part, toute contrefaçon, plagiat, reproduction illicite encourt une poursuite pénale.

Contact bibliothèque : [ddoc-theses-contact@univ-lorraine.fr](mailto:ddoc-theses-contact@univ-lorraine.fr)  
*(Cette adresse ne permet pas de contacter les auteurs)*

## LIENS

Code de la Propriété Intellectuelle. articles L 122. 4

Code de la Propriété Intellectuelle. articles L 335.2- L 335.10

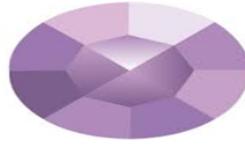
[http://www.cfcopies.com/V2/leg/leg\\_droi.php](http://www.cfcopies.com/V2/leg/leg_droi.php)

<http://www.culture.gouv.fr/culture/infos-pratiques/droits/protection.htm>



UNIVERSITÉ  
DE LORRAINE

IAEM



---

Thèse de doctorat de l'Université de Lorraine  
Ecole Doctorale : IAEM  
Mention Automatique, Traitement du Signal et des Images

---

# Observer Design for Nonlinear Systems using LMI Relaxation Techniques

---

*Présentée par : Shivaraj Mohite*  
*Date de Soutenance: 20 Décembre 2023*

Jury :

Direction de thèse:	M. Ali ZEMOUCHE	Maître de Conférences, HDR, Université de Lorraine
CoDirection de thèse:	M. Marouane ALMA	Maître de Conférences, Université de Lorraine
Rapporteur:	Mme. Patrizia BAGNERINI	Associate Professor, Université de Gènes
Rapporteur:	Mme. Taous-Meriem LALEG-KIRATI	Chargée de recherche, HDR, INRIA
Examineurs:	Mme. Catherine BONNET	Directrice de recherche, INRIA
	M. Dalil ICHALAL (Président du jury)	Professeur des Universités, Université d'Evry
	M. Fan ZHANG	Associate Professor, Université Sun Yat-Sen
	M. Michel ZASADZINSKI	Professeur des Universités, Université de Lorraine
Invité:	M. Mohamed ZERROUGUI	Maître de Conférences, Université Aix-Marseille



## Abstract

Over the last three decades, observer design for dynamical systems has emerged as a pivotal topic in the control system domain. This is a consequence of the fact that observers are used to collect real-time information about the systems, monitor the systems, control the system, and make decisions. The LMI-based observers have garnered considerable attention among the various established observer methodologies. The thesis predominantly deals with the design of the LMI-based observers for the Lipschitz nonlinear systems. One of the valuable contributions of this thesis is the establishment of more generalized matrix-multiplier-based LMIs. These LMI conditions are derived through the utilization of reformulated Lipschitz property, Young inequalities, and newly defined matrix multipliers. Further, an LMI-based  $\mathcal{H}_\infty$  observer is designed to estimate the states of the nonlinear systems under the presence of disturbances/noise by using the previously defined matrix multipliers. The proposed LMIs contain additional numbers of decision variables as compared to the existing approaches, which add extra degrees of freedom thus improving their feasibility. The effectiveness of the developed approach is validated through numerical examples.

This new matrix-multiplier-based LMI approach is extended for the state estimation for a class of non-globally Lipschitz nonlinear systems under the effect of noise/disturbances. It is achieved by deploying the Hilbert projection-based observer structure. The novel LMI condition is established to ensure the ISS behaviour of the state estimation error of the proposed observers. Furthermore, the proposed matrix-multiplier-based LMI approach is exploited for the stabilization of the disturbance-affected nonlinear systems using the observer-based controller. The judicious implementation of the matrix-multiplier-based LMIs made it possible to obtain less conservative LMI conditions than existing LMI methods in the literature. Finally, all these developed LMI-based observers are used in the autonomous vehicle area for various tasks, such as longitudinal state estimation, slip angle estimation, and so on to validate their performances.

Keywords: Nonlinear observer design, Lipschitz systems, Lyapunov stability, Input-to-state stability and Linear Matrix Inequalities (LMIs).

## Résumé

Au cours des trois dernières décennies, la conception d'observateurs pour les systèmes dynamiques est devenue un sujet important dans le domaine des systèmes de contrôle. Ceci est dû au fait qu'un observateur est utilisé en temps-réel pour la supervision des systèmes, le contrôle des systèmes, ainsi que pour la prise de décision. Les observateurs d'état basés sur les techniques LMIs ont suscité une attention par rapport aux autres techniques d'observation. Cette thèse de doctorat porte sur les techniques de relaxation LMIs pour des classes de systèmes non linéaires avec des non-linéarités lipschitziennes. La contribution principale de la thèse est la proposition de nouvelles matrices dites *Matrix-Multiplier*. Les conditions LMIs obtenues en utilisant ces matrices (ainsi qu'une reformulation de la propriété de Lipschitz et la relation de Young de façon judicieuse) ont été testées moins contraignantes que les autres techniques existantes dans la littérature. Une extension au cas  $\mathcal{H}_\infty$  a été proposée pour les systèmes en présence de bruits ou perturbations. Grace aux variables de décisions supplémentaires créées par les *Matrix-Multiplier* permettent d'améliorer la faisabilité des conditions LMIs. L'efficacité et la supériorité de la nouvelle technique ont été validées à travers plusieurs exemples illustratifs.

Une version de la nouvelle approche LMI proposée est étendue à une classe de systèmes non linéaires avec des non-linéarités localement lipschitziennes, en présence des bruits/perturbations. Cela est réalisé en utilisant la structure d'observateur basée sur la projection de Hilbert. La nouvelle condition LMI est établie pour garantir le comportement ISS de l'erreur d'estimation d'état des observateurs proposés. De plus, l'approche LMI basée sur les multiplicateurs de matrices proposée est exploitée pour la stabilisation des systèmes non linéaires affectés par des perturbations en utilisant une commande basée observateurs. La mise en œuvre judicieuse des LMI basés sur les multiplicateurs de matrices a permis d'obtenir des conditions LMI moins conservatrices que les méthodes LMI existantes dans la littérature. Enfin, tous ces observateurs basés sur les LMI développés sont utilisés dans le domaine des véhicules autonomes pour diverses tâches, telles que l'estimation de l'état longitudinal, l'estimation de l'angle de glissement, etc., afin de valider leurs performances.

Mots- clés : observateurs ; inégalités matricielles linéaires (LMIs) ; stabilité de Lyapunov.

## Acknowledgments

*“The most groundbreaking discoveries are born from the collective wisdom of researchers working in harmony.”*

---

*Neil deGrasse Tyson*

I am delighted to express my heartfelt gratitude to my supervisors, Dr Ali Zemouche and Dr Maraoune Alma, for providing me with the opportunity to conduct this research under their supervision, as well as for their enduring support and encouragement throughout my PhD candidature at the University of Lorraine, France.

I thank Dr N. M. Singh and Dr S. R. Wagh associated with Control and Decision Research Center (CDRC), EED, VJTI, Mumbai, India, for providing technical guidance and support during this PhD journey. I would like to express my genuine gratitude to Mr Naim Bajcinca associated with MEC, RPTU, Germany, for their technical guidance.

Further, I am thankful for the support I have received from my fellow labmates, Hichem, Ayoub, Echrak, Sofiane and Hasni during my time in the lab. In addition to this, I would like to say thanks to my best friends, Dyhia, Loris Simon and Mohamed Laaziri, as well as my previous co-author, Dr Adil Sheikh for their constant companionship during my PhD journey. Lastly, I would like to acknowledge Mr. J (My heart’s desire) for being an emotional pillar during this important phase of my life.



*I dedicate this thesis to three beloved individuals who have been integral parts of my life. First and foremost, to my paternal grandfather, Yashwant Mohite, whose love for me knew no bounds and who taught me the value of hard work. Thank you so much, Appa; I will never forget you.*

*Although he is no longer in this world, his memories continue to influence my life.*

*Next, I dedicate this thesis to my mother, Mrs. Surekha Mohite, who raised me with love and whose dream I am fulfilling now. She has always supported me throughout my life, and I am confident she will continue to support me in the future. I can never repay her sacrifices for my education, but this dedication is a small token of my gratitude.*

*Last but not least, I dedicate this thesis to my soulmate, 'Mr. J'. You are the only person in my life, after my mom, who has continuously stood by me through all my ups and downs, especially during this PhD journey.*

*I love you all.*





---

# Contents

List of Figures	xiii
List of Tables	xv
Notations and Acronyms	xvii
Introduction	

<b>Chapter 1</b>	
<b>Nonlinear Observer Design: Fundamentals and Methodologies</b>	
1.1 Introduction . . . . .	7
1.2 Some important notions . . . . .	9
1.2.1 State observer . . . . .	9
1.2.2 Observability . . . . .	11
1.2.2.1 Geometric conditions . . . . .	11
1.2.2.2 Analytic conditions . . . . .	14
1.2.3 Stability . . . . .	16
1.2.4 Unveiling Lyapunov theory: a comprehensive study of system stability . .	18
1.2.5 Reiterating ISS notion: grasping its Impact on the system stability . . . .	21
1.3 State-of-the-art on nonlinear observer design . . . . .	22
1.3.1 Nonlinear transformation based method . . . . .	23
1.3.2 High-gain observers . . . . .	25

1.3.3	Luenberger observers . . . . .	27
1.3.3.1	LMI-based approach . . . . .	27
1.3.3.2	Riccati equation-based method . . . . .	28
1.3.4	Revealing the nonlinear observer design for the disturbance-affected systems . . . . .	29
1.3.4.1	ISS based approach . . . . .	29
1.3.4.2	$\mathcal{H}_\infty$ observer . . . . .	30
1.3.5	A generalised Arcak's observer . . . . .	32
1.4	Conclusion . . . . .	33

## Chapter 2

### Evolving the LMI-based nonlinear observers: matrix multipliers as a key element

2.1	Introduction . . . . .	36
2.2	Towards enhanced LMI-based observers: hurdles and potentialities . . . . .	36
2.3	Preliminaries . . . . .	39
2.4	Incorporating the matrix multipliers in LMI-based nonlinear observers . . . . .	42
2.4.1	Problem formulation . . . . .	42
2.4.2	Synthesis of LMI condition . . . . .	45
2.4.3	Comment about the proposed LMI condition . . . . .	47
2.5	Enhancement in the matrix multiplier-based LMI approach . . . . .	47
2.5.1	Problem Contextualization . . . . .	47
2.5.2	LMI design with Young inequality . . . . .	50
2.5.3	LPV-based LMI approach . . . . .	52
2.5.4	LMIs for the nonlinear systems with linear outputs . . . . .	53
2.6	Validation of proposed methodologies . . . . .	54
2.6.1	Numerical example 1 . . . . .	54
2.6.2	Numerical example 2 . . . . .	55
2.6.3	Implementation on a robotic arm manipulator . . . . .	56
2.7	Conclusion . . . . .	58

## Chapter 3

### Observers for the disturbance-affected nonlinear systems: an enhanced LMI approach

3.1	Introduction . . . . .	60
3.2	Design of $\mathcal{H}_\infty$ nonlinear observer . . . . .	61
3.2.1	Articulating the problem statement . . . . .	61
3.2.2	LMI formulation with Young standard inequality . . . . .	65

3.2.3	An Enhanced LMI criterion: exploiting LPV approach . . . . .	67
3.2.4	$\mathcal{H}_\infty$ observers for the nonlinear systems with linear outputs . . . . .	68
3.3	LMI-based observers for locally Lipschitz systems: exploiting an ISS framework . . . . .	70
3.3.1	Insight on the Hilbert projection . . . . .	70
3.3.2	ISS-based observer design: articulating the problem statement . . . . .	72
3.3.3	LMI synthesis . . . . .	75
3.3.3.1	Formulating the essential conditions for ISS . . . . .	75
3.3.3.2	Matrix-multipliers-based LMI formulation . . . . .	78
3.3.4	Observer design for systems having nonlinear dynamics and linear outputs . . . . .	80
3.4	Analysing the effectiveness of matrix-multipliers-based LMIs . . . . .	81
3.4.1	Number of decision variables . . . . .	81
3.4.2	A comparison of the number of additional decision variables in different matrix multiplier structures . . . . .	82
3.5	Evaluation of the proposed LMIs and the observers . . . . .	85
3.5.1	Numerical Example 1 . . . . .	85
3.5.2	Numerical Example 2 . . . . .	89
3.6	Conclusion . . . . .	92

## Chapter 4

### Nonlinear Observer-based stabilisation using new matrix multiplier-based LMIs

4.1	Introduction . . . . .	93
4.2	Contextualising the problem statement . . . . .	94
4.3	A robust circle-criterion based approach . . . . .	97
4.4	ISS-based LMI formulation . . . . .	101
4.5	Validating the performance of the developed methodology . . . . .	104
4.5.1	Example 1: A single-link flexible joint robot model . . . . .	104
4.5.2	Example 2: A numerical example of unstable system . . . . .	106
4.6	Conclusion . . . . .	109

## Chapter 5

### Implementing the proposed observers in autonomous vehicle applications

5.1	Introduction . . . . .	111
5.2	state estimation of the longitudinal autonomous vehicle model . . . . .	112
5.2.1	Longitudinal vehicle dynamics . . . . .	112
5.2.2	Observer-based state estimation . . . . .	114
5.3	Slip angle estimation using ISS-based observer . . . . .	117
5.3.1	Lateral dynamics of vehicle . . . . .	117

5.3.2	Employing ISS-based observer . . . . .	118
5.4	Transformation-based observer for vehicle tracking model . . . . .	120
5.4.1	Vehicle model . . . . .	121
5.4.2	Observer development . . . . .	121
5.5	Conclusion . . . . .	125
<b>Conclusion and perspectives</b>		<b>127</b>
<b>Appendix A</b>		
<b>Tools used in nonlinear observer design</b>		
A.1	Lipschitz property . . . . .	129
A.2	Young Relation (Completion of the Squares) . . . . .	130
A.3	Schur lemma . . . . .	131
A.4	Linear Matrix Inequalities . . . . .	131
<b>Bibliography</b>		<b>133</b>
<b>Introduction</b>		<b>143</b>
<b>Conclusion et perspectives</b>		<b>147</b>

# List of Figures

1.1	Illustrating the obstacles associated with sensors . . . . .	8
1.2	Tasks performed by an observer . . . . .	8
1.3	Illustration of an observer as an auxiliary system . . . . .	10
1.4	Schematic representation of geometric conditions of observability . . . . .	12
1.5	Various observer design methods . . . . .	23
1.6	Diagrammatic representation of Arcaç's observer . . . . .	33
2.1	Encouraging towards the development of new matrix multiplier-based LMIs . . . . .	40
2.2	Plot of estimation error ( $\tilde{x}$ ) in <b>Case 1</b> . . . . .	56
2.3	Plot of estimation error ( $\tilde{x}$ ) in <b>Case 2</b> . . . . .	56
2.4	Plot of estimation error ( $\tilde{x}$ ) in <b>Case 3</b> . . . . .	57
3.1	Visualisation of disturbance-affected nonlinear system . . . . .	60
3.2	Depiction of Hilbert projection . . . . .	71
3.3	Plot of estimation error dynamic ( $\tilde{x}$ ) in <b>Case 1</b> . . . . .	88
3.4	Plot of estimation error dynamic ( $\tilde{x}$ ) in <b>Case 2</b> . . . . .	88
3.5	Plot of estimation error dynamic ( $\tilde{x}$ ) in <b>Case 3</b> . . . . .	88
3.6	Plot of the estimation error ( $\tilde{x}$ ) in <b>Case 1</b> . . . . .	90
3.7	Plot of the estimation error ( $\tilde{x}$ ) in <b>Case 2</b> . . . . .	91
3.8	Plot of the estimation error ( $\tilde{x}$ ) in <b>Case 3</b> . . . . .	92
4.1	Illustration of observer-based controller . . . . .	95
4.2	Graphical representation of Estimation error ( $\tilde{x}$ ) in case of the model of a single-link joint robot . . . . .	106
4.3	Graphical presentation of stabilised states ( $\tilde{x}$ ) for the application of a single-link robot . . . . .	106
4.4	Visualisation of unstable states . . . . .	107
4.5	Representation of stabilised states . . . . .	108

---

4.6	Plot of estimation error ( $\tilde{x}$ ) . . . . .	108
5.1	Input $u(t)$ given to nonlinear vehicle model . . . . .	115
5.2	Trajectories of estimation error ( $\tilde{x}$ ) in several cases . . . . .	116
5.3	Pictorial representation of a single track model for lateral vehicle dynamic . . . . .	117
5.4	Estimation of Slip angle . . . . .	119
5.5	Graph of estimation error ( $\tilde{x}$ ) . . . . .	119
5.6	Vehicle motion model . . . . .	120
5.7	Graphical depiction of the proposed transformation-based observer . . . . .	122
5.8	Estimation of relative longitudinal position ( $X$ ) of a vehicle . . . . .	124
5.9	Estimation of relative lateral position ( $Y$ ) of a vehicle . . . . .	124
5.10	Estimation of yaw angle ( $\psi$ ) of a vehicle . . . . .	124

# List of Tables

1.1	Summary of the analytic conditions of observability . . . . .	16
1.2	Summary of HGO approaches . . . . .	26
2.1	Summary of LMI-based nonlinear observer approaches . . . . .	36
2.2	Computation of the number of decision variables of LMI (2.55) . . . . .	46
2.3	Number of additional variables for several LMI methods . . . . .	47
2.4	Maximum value of $\theta$ (i.e., $\theta_m$ ) for which LMI (2.55) is feasible . . . . .	54
2.5	Comparison of different LMI methods . . . . .	56
2.6	Representation of the results obtained from the solution of the proposed LMIs . . . . .	57
3.1	A quantitative analysis of the variables inside different matrix configurations . . . . .	84
3.2	Optimal values of $\sqrt{\mu}$ for different cases . . . . .	87
3.3	Values of gain matrix $L$ for different cases . . . . .	87
3.4	Compiling the optimal values obtained from LMI solution . . . . .	91
3.5	Comparison of RMSE values of $\tilde{x}$ in different cases . . . . .	91
4.1	Parameters of the model of a single-link flexible joint robot . . . . .	105
4.2	Comparison results for different values of $m$ . . . . .	105
4.3	Recapitulating solutions obtained from different LMIs . . . . .	107
5.1	The models to depict the vehicular longitudinal motion . . . . .	113
5.2	Vehicle parameters . . . . .	114
5.3	Obtained optimal values in different cases . . . . .	116
5.4	Parameters of vehicle model . . . . .	119



# Notations and Acronyms

## Notations

### Sets, Norms and Vectors

$\mathbb{Z}^+$	Set of positive integers, i.e., $\{1, 2, \dots\}$
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^n$	$n$ -dimensional Euclidean real space
$\mathbb{R}^{n \times m}$	Set of $n \times m$ dimensional real matrices
$\mathbf{S}^n$	Set of $n \times n$ dimensional real symmetric matrices
$x_0$	Initial value of $x(t)$ at $t = 0$
$\ x\ $	Euclidean norm of $x \in \mathbb{R}^n$
$\ x\ _{\mathcal{L}_2}$	$\mathcal{L}_2$ norm of $x \in \mathbb{R}^n$
$\text{sgn}(a)$	Signum function of a real number $a$
$e_n(i)$	A $n$ dimensional vector which has unity at $i^{\text{th}}$ position and zero elsewhere, i.e., $\underbrace{(0, \dots, 0, \overset{i^{\text{th}}}{1}, 0, \dots, 0)}_{n \text{ components}}^\top$

### Matrix notations

$\mathbb{I}_n$	Identity matrix of order $n \times n$
$\mathbb{O}_n$	Null matrix of order $n \times n$
$A = \text{diag}(a_1 \ \cdots \ a_n)$	Diagonal matrix of order $n \times n$ whose diagonal elements are $a_1, \dots, a_n$
$A = \text{block-diag}(A_1 \ \cdots \ A_n)$	Block-diagonal matrix of order $n \times n$ whose diagonal elements are $A_1, \dots, A_n \in \mathbb{R}^{\bar{n} \times \bar{n}}$
$\star$	It represents repeated blocks within a symmetric matrix
$A_{i,j}$	Entry of matrix $A$ in $i^{\text{th}}$ row and $j^{\text{th}}$ column

$A > 0$	Symmetric positive definite matrix
$A < 0$	Symmetric negative definite matrix
$A \geq 0$	Symmetric semi-positive definite matrix
$A \leq 0$	Symmetric negative semi-definite matrix
$\det(A)$	Determinant of a matrix $A$
$\lambda_{\max}(A)$	Maximum eigenvalue of a matrix $A$
$\lambda_{\min}(A)$	Minimum eigenvalue of a matrix $A$

## Abbreviation

LMI	Linear Matrix Inequality
w.r.t.	with respect to
ISS	Input-to-state stability
ARE	Algebraic Riccati Equation
LPV	Linear parameter varying
$\delta$ QC	Incremental Quadratic Constraints
DMVT	Differential Mean Value Theorem
DOF	degrees of freedom

# List of publications

- **International Journals**

1. **Shivaraj Mohite**, Alma Marouane, Ali Zemouche (2023). “Observer-Based Stabilization of Lipschitz Nonlinear Systems by Using a New Matrix-Multiplier Based LMI Approach”, *IEEE Control Systems Letters*.
2. **Shivaraj Mohite**, Alma Marouane, Ali Zemouche (2023). “Towards enhancing nonlinear observers for the Lipschitz system: exploiting the matrix multipliers-based LMIs”, *International Journal of Robust Nonlinear Control*. (**Submitted**)

- **International Conferences**

1. **Shivaraj Mohite**, Alma Marouane, and Ali Zemouche (2023). “An LMI-Based Observer Design Method for a Class of Nonlinear Systems ”, *22<sup>nd</sup> European Control Conference (ECC)* , Stockholm, Sweden. (**Submitted**)
2. **Shivaraj Mohite**, Alma Marouane and Ali Zemouche (2023). “Design of a Nonlinear Observer for a Class of Locally Lipschitz Systems by Using Input-To-State Stability: An LMI Approach ”, *62<sup>nd</sup> IEEE Conference on Decision and Control*, Singapore.
3. **Shivaraj Mohite**, Alma Marouane, Haddad Madjid and Ali Zemouche (2023). “State Estimation of Longitudinal Vehicle Model Using  $\mathcal{H}_\infty$  LMI-Based Nonlinear Observer ”, *31<sup>st</sup> Mediterranean Conference on Control and Automation (MED 2023)*, Limassol, Cyprus.
4. **Shivaraj Mohite**, Alma Marouane, Haddad Madjid and Ali Zemouche (2023). “LMI-Based  $\mathcal{H}_\infty$  Observer Design for Nonlinear Lipschitz System ”, *22<sup>nd</sup> World Congress of the IFAC* , Yokohama, JAPAN.
5. **Shivaraj Mohite**, Alma Marouane and Ali Zemouche (2022). “A New Matrix Multiplier-Based LMI Approach for Nonlinear Observers”, *30<sup>th</sup> Mediterranean Conference on Control and Automation (MED 2022)*, Athens, Greece.

6. **Shivaraj Mohite**, Ali Zemouche, Haddad Madjid, Alma Marouane, Delattre Cédric, and N.M. Singh (2021). “ $\mathcal{H}_\infty$  switched-gain based observer vs nonlinear transformation based observer for a vehicle tracking model”, *3<sup>rd</sup> IFAC Conference on Modelling, Identification and Control of Nonlinear Systems (MICNON) 2021*, Tokyo, Japan.



---

# Introduction

*“In the discipline of control system engineering, nonlinear observers emerge as reliable tools, revealing the intricate dynamics of nonlinear complex systems to enhance state estimation strategies.”*

---

*Unknown*

## Background and Scope of the research

State estimation of dynamical systems is a pivotal research topic in the control system. The knowledge of such real-time information of the systems obtained from the state estimation is used for tasks related to these systems such as monitoring, controlling, and decision-making. Thus, these tools play a vital role in various applications, such as self-synchronization in multi-agent systems [1]; cyber-attacks and faults detection in the connected autonomous vehicle platoon [2], and so on. In the literature, numerous methodologies have been developed for the purpose of the state estimation of the linear systems, for example, Kalman filters [3], Luenberger observer [4]. However, the design of such tools for nonlinear systems is still a challenging task. This is due to the complex structure of system nonlinearities. Several approaches have been proposed in the literature to reconstruct the states of nonlinear systems, for instance, the extended Kalman filter [5], the unscented Kalman filter [6], the high-gain observers [7], the sliding-mode observers [8], etc. In addition to these techniques, the LMI-based nonlinear observers have recently received a lot of attention in the control domain [9–12]. Despite various effective LMI-based observer design strategies presented in contemporary literature, there is a potential for further improvements in the LMI-based approaches. These possibilities of enhancements in LMI-based methodologies inspire the author to carry out their research work in this domain. The subsequent segments illustrate the primary focus and the detailed framework of this thesis.

## Objective

As mentioned earlier, the LMI-based nonlinear observer design approach has become one of the trending research topics in control system engineering. Several LMI-based observer techniques are presented in [12–17]. Some of these approaches depended on S-Procedure lemma [18], Riccati equations [19] and Young inequality [20]. All these methods provide an enhanced LMI condition as compared to the other approaches. Recently, some enhanced LMI conditions have been showcased in [14, 15, 21, 22]. Though all these LMI methods are enhanced and less conservative

than the existing ones, there is scope for further improvements. Among all these aforementioned methodologies, the approach proposed in [14] is based on the reformulated Lipschitz property of the nonlinearities and a newly defined variant of Young inequality. One of the fundamental components of this method is the introduction of a block-diagonal matrix multiplier which was inspired by [12, 13]. The authors of [14] papers provided enhanced LMI conditions with additional numbers of degrees of freedom from a feasibility point of view. The addition of a block-diagonal multiplier matrix in the LMI approach is intriguing and beneficial, yet there is still scope for improvement. The question that arises here is why a block-diagonal matrix structure only. This question inspires the authors to propose a novel LMI condition based on a more generalised matrix multiplier than a block diagonal matrix. The inclusion of such a matrix multiplier inside LMIs can be beneficial in terms of the enhancement of existing ones. In the summarised manner, this thesis primarily concentrates on the development of novel LMI conditions that have more degrees of freedom and a larger feasibility region than those presented in [13, 14, 22]. The objectives of the research work presented in this thesis are outlined as follows:

- **To design a new LMI-based observer design:** The key goal is to propose an observer design approach for the class of globally nonlinear Lipschitz systems by utilising the new LMI conditions, which are less conservative than the existing approach, and offer more enhancements from a feasibility point of view.
- **To employ the proposed LMI approach for several control applications:** In the second part of this thesis, the authors aim to implement the previously developed matrix-multiplier-based LMI approach for estimating the states of nonlinear systems affected by disturbances. Further, the thesis focuses on designing nonlinear observers tailored for non-global Lipschitz systems. Later on, this thesis will delve into the topic of the observer-based stabilization approach for nonlinear systems using LMIs.
- **To implement the developed observer techniques in autonomous vehicle technology:** The purpose of the last segment of this thesis is to deploy the proposed nonlinear observers into various autonomous vehicle tasks.

In the sequel, we showcased the layout of this thesis.

## Thesis outline

The elaborated layout of the thesis is described in the following manner:

- I) An overview of some important notions required in observer design and several existing observer techniques are presented in **Chapter 1**. In order to enhance comprehensibility, **Chapter 1** is structured as follows: Section 1.2 is dedicated to the overview of some fundamental concepts that are essential for the nonlinear observer design. Section 1.3 provides a summary of various existing nonlinear observer design methodologies. Further, Section 1.4 contains some conclusions.
- II) **Chapter 2** is dedicated to the introduction of novel matrix-multiplier-based LMI conditions which are less conservative than the existing ones. The authors have organised this chapter in a subsequent way: An overview of LMI-based observer and the motivation of the proposed methodology is discussed in Section 2.2. The prerequisites required for new observer design methods are illustrated in Section 2.3. A new matrix multiplier-based LMI approach is presented in 2.4. Further, Section 2.5 contains an enhanced matrix

multiplier-based LMI approach. Section 2.5 includes the validation of the proposed techniques through numerical examples. In Section 2.7, some conclusions are given.

- III) In next parts of this thesis, **Chapter 3** encompasses the implementation of the proposed new LMI-based observer techniques for several tasks, such as the state estimation of the disturbance-affected nonlinear system and observer design for the non-globally Lipschitz system. The configuration of this chapter is illustrated as: Section 3.2 encompasses the design of  $\mathcal{H}_\infty$  nonlinear observer. The development of the LMI condition based on standard Young inequality is illustrated in Subsection 3.2.1, whereas Subsection 3.2.2 includes the formulation of the LPV-based LMI approach. The extension of the proposed new matrix-multiplier based-LMI approach for the nonlinear system with local Lipschitz nonlinearities is showcased in Section 3.3. Further, Section 3.4 emphasises the effect of the different structures of matrix multipliers on the proposed LMI. The enhancement of the developed LMI conditions and the validation of the proposed observer is demonstrated in section 3.5. Some conclusions are drawn in Section 3.6.
- IV) Further, in **Chapter 4**, an observer-based stabilization strategy for nonlinear systems is established by using our newly developed matrix multiplier-based LMI approach. We arranged this chapter in the following manner: Section 4.2 focuses on the formulation of the problem statement. For the proposed observer-based controller, a novel LMI condition is formulated in Section 4.3 using the  $\mathcal{H}_\infty$  criterion. Further, the property of ISS is utilised in Section 4.4 for deriving an LMI criterion. The performance of the proposed approaches is validated in Section 4.5 through numerical examples. In the last, Section 4.6 provides some concluding remarks.
- V) Later on, the deployment of the proposed observer in the domain of autonomous vehicles is showcased in **Chapter 5**. The layout of this chapter is illustrated as follows: Section 5.2 is dedicated to implementing the proposed observers of Chapter 3 on the longitudinal vehicle model. The deployment of the ISS-based observer of Chapter 3 for the purpose of slip-angle estimation of the nonlinear vehicle model is illustrated in Section 5.3. Further, Section 5.4 encompasses the development of the coordinate transformation-based nonlinear observer for a vehicle tracking model. Finally, some conclusions are showcased in Section 5.5.

In the next segment, the authors illustrate the contributions of the research work presented in this thesis.

## Contribution of the thesis

The overall contribution of this thesis is listed as:

- A. **The establishment of new and enhanced LMI condition:** One of the notable outcomes of this thesis is the development of novel LMI conditions based on a matrix multiplier. These conditions are formulated by integrating the newly defined matrix multiplier with a well-established LPV approach and a recently introduced reformulated Lipschitz property. Due to the judicious use of the newly proposed matrix multiplier, the obtained LMI conditions contain extra numbers of decision variables than the existing LMI approaches [12–14]. The additional decision variables expanded the LMI’s feasibility region, enhancing these conditions from a feasibility point of view.

- B. **The applicability of the proposed LMI method:** The developed LMI approach is applied in various control tasks, as described below:
- 1) The developed matrix-multiplier-based LMI approach is employed to design an enhanced  $\mathcal{H}_\infty$  observer for estimating the state of nonlinear systems affected by external disturbances and noise in both their dynamics and outputs. The established LMI conditions provide a better noise attenuation than the existing methods of [12–14], as shown in numerical examples.
  - 2) Further, for the disturbance-affected non-global Lipschitz nonlinear systems, the observer proposed in [23] is incorporated with the newly developed LMI approach for state estimation. The obtained LMI conditions are formulated by using the ISS notion. The utilisation of this developed method relaxes the main assumption of global Lipschitz nonlinearities required during nonlinear observer design problems.
  - 3) Later on, the new LMI condition is designed for observer-based stabilisation of a class of nonlinear systems by integrating newly defined matrix multipliers, reformulated Lipschitz property, and Young inequalities. Due to the deliberate deployment of these mathematical tools in the LMI formulation, the resulting conditions are enhanced and less conservative than the one presented in [24–26]
- C. **The deployment of the proposed observers in several autonomous vehicle applications:** The formulated LMI-based observer methodology is deployed for various autonomous vehicle tasks, including longitudinal state estimation, vehicle tracking, and slip angle estimation. It aids in validating the performance of the proposed observer in practical applications.

---

# Nonlinear Observer Design: Fundamentals and Methodologies

---

## Contents

<b>1.1</b>	<b>Introduction</b>	<b>7</b>
<b>1.2</b>	<b>Some important notions</b>	<b>9</b>
1.2.1	State observer	9
1.2.2	Observability	11
1.2.3	Stability	16
1.2.4	Unveiling Lyapunov theory: a comprehensive study of system stability	18
1.2.5	Reiterating ISS notion: grasping its Impact on the system stability	21
<b>1.3</b>	<b>State-of-the-art on nonlinear observer design</b>	<b>22</b>
1.3.1	Nonlinear transformation based method	23
1.3.2	High-gain observers	25
1.3.3	Luenberger observers	27
1.3.4	Revealing the nonlinear observer design for the disturbance-affected systems	29
1.3.5	A generalised Arcak's observer	32
<b>1.4</b>	<b>Conclusion</b>	<b>33</b>

---

## 1.1 Introduction

One of the fundamental topics in control system engineering is observer design. Since observers play a vital role in state estimation and control, it has attracted a lot of attention. This is mainly due to the fact that knowing the current state of the systems is required in many applications. In practical situations, this real-time information is necessary for monitoring, decision-making, and controlling the systems. In such cases, the use of sensors is possible in order to obtain information from the system. However, in reality, not all states are measurable, either because of

the expensive cost of sensors or simply because of their unavailability. These problems associated with the sensors are summarised in Figure 1.1.

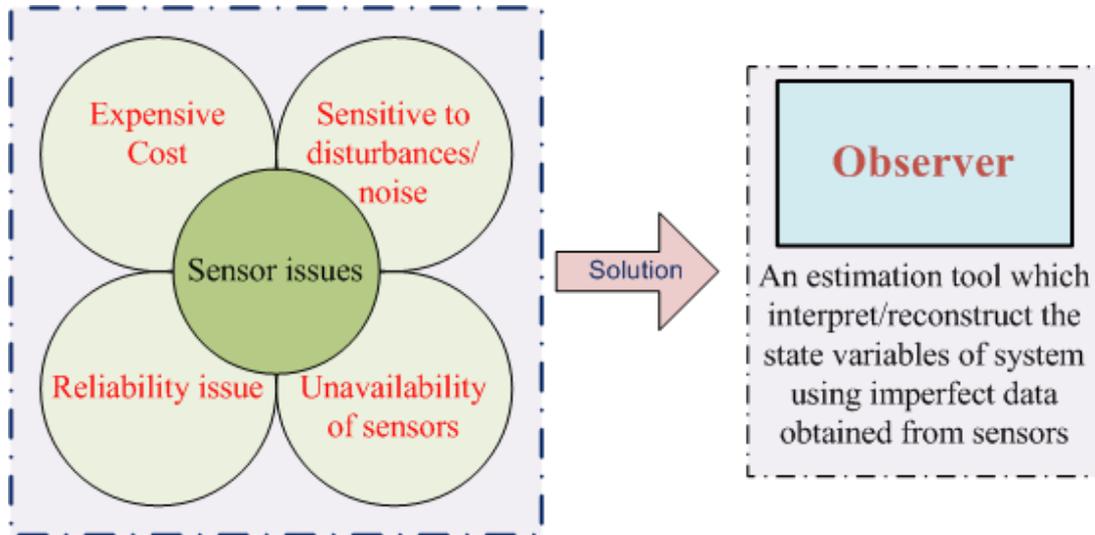


Figure 1.1: Illustrating the obstacles associated with sensors

An alternative for the sensors is to use an observer for the reconstruction of the states of the system. In addition to this, the observer structure is employed for estimating unknown

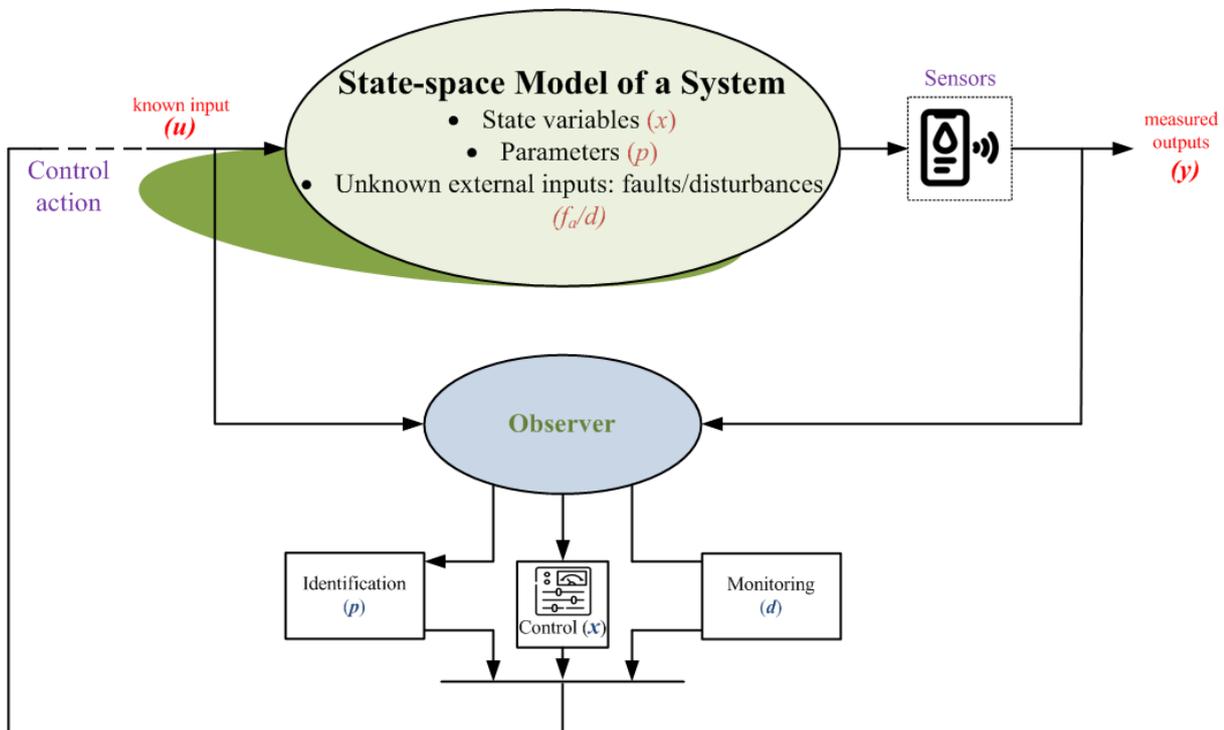


Figure 1.2: Tasks performed by an observer

parameters of systems, unknown external signals (for example, faults or cyber-attacks), and so on. Figure 1.2 illustrates the various tasks performed by the observers. Hence, the observer is an important element in the control system domain.

The objective of this chapter is to provide an overview of some important notions and several existing observer techniques. This chapter is arranged as follows: Section 1.2 encompasses a summary of some important concepts that are essential for the nonlinear observer design. Within this section, the definitions of the observer and observability are provided in Subsections 1.2.1 and 1.2.2, respectively. Whereas Subsections 1.2.3, 1.2.4 and 1.2.5 contain the overview of stability conditions, Lyapunov conditions from the stability point of view, and ISS property, respectively. Section 1.3 is dedicated to a synopsis of various existing nonlinear observer design methodologies. Further, some conclusions are drawn in Section 1.4.

## 1.2 Some important notions

This section provides an outline of some fundamental concepts such as the observer, observability, and stability of the system. The aforementioned concepts enable us to better comprehend how observer design performs.

### 1.2.1 State observer

Let us consider a generalised class of nonlinear systems as follows:

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)), \\ y(t) &= h(x(t)),\end{aligned}\tag{1.1}$$

where

- $x \in \mathbb{R}^n$  is a state vector, taking values in a connected  $n$  dimensional manifold  $X$ .
- $u \in \mathbb{R}^s$  denotes the external input vector, which takes values in some  $s$  dimensional open subset  $U$ . Input functions  $u(\cdot)$  are assumed to be locally bounded and measurable in a set  $\mathcal{U}$ .
- The vector  $y \in \mathbb{R}^p$  indicates measured outputs which are confined in some  $p$  dimensional open subset  $Y$ .
- Functions  $f$  and  $h$  represent the nonlinearities present in the system dynamics and outputs, respectively. Further, both are assumed to be Lipschitz-continuous w.r.t. their arguments.

Let  $\Phi_u(t, x_0)$  denotes the solution of (1.1) by applying input  $u(t)$  on  $[0, t]$  with an initial condition  $\Phi_u(x_0) = x_0$ . In order to control or monitor the system (1.1), it is necessary to know  $x(t)$ , but in practical cases, the available measurements are restricted to  $y$  or  $u$ . The observation problem is formulated as:

Find an estimate  $\hat{x}(t)$  for  $x(t)$  for a given system (1.1) using the information of  $u(t_1)$  and  $y(t_1)$  for  $0 \leq t_1 \leq t$ .

This problem makes sense when  $h$  w.r.t.  $x$  cannot be inverted at any time.

One of the possible solutions to this problem is to use optimization tools. In terms of optimisation, one can look for the best estimate  $\hat{x}_0$  of  $x_0$  which provides  $y(t_1)$  over  $0 \leq t_1 \leq t$ . Further, by

integrating (1.1) from  $\hat{x}_0$  and the knowledge of  $u(t_1)$ , it is easy to obtain  $\hat{x}(t)$ . For more details, one can refer to [27] and [28]. Though this approach provides a systematic way to find  $\hat{x}(t)$ , it has some drawbacks such as computational burden, local minima, and so on.

An alternative method is to use the concept of explicit feedback in the estimation of  $x(t)$ . In a simple manner, one can obtain the estimate of  $x(t)$  by integrating (1.1) from  $x_0$  if  $x_0$  is known. When  $x_0$  is unknown, one can try to integrate (1.1) for some erroneous  $\hat{x}_0$  to determine  $\hat{x}$ . Further, the estimated  $\hat{x}(t)$  can be corrected online according to the error  $h(\hat{x}(t)) - y(t)$ . In another words, one can find an estimate  $\hat{x}$  of  $x$  as a solution of the following system:

$$\dot{\hat{x}}(t) = f(\hat{x}(t), u(t)) + \kappa(t, h(\hat{x}(t)) - y(t)), \text{ with } \kappa(t, 0) = 0. \quad (1.2)$$

Such an auxiliary system is known as an observer, and (1.2) represents the standard form of an observer for a system (1.1). The schematic representation of an observer (1.3) for the system (1.1) is shown in Figure 1.3.

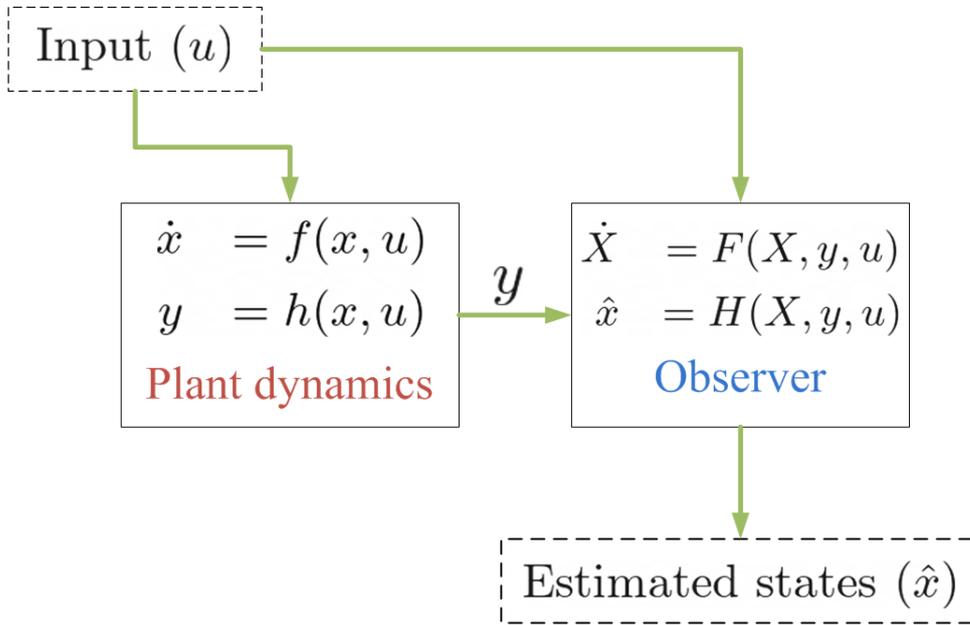


Figure 1.3: Illustration of an observer as an auxiliary system

A more generalised definition of an observer is as follows:

**Definition 1.1: Observer [29]:**

For a system (1.1), an observer is given by an auxiliary system:

$$\begin{aligned} \dot{X}(t) &= F(X(t), u(t), y(t), t) \\ \hat{x}(t) &= H(X(t), u(t), y(t), t) \end{aligned} \quad (1.3)$$

such that

- (i)  $\hat{x}(0) = x(0) \Rightarrow \hat{x}(t) = x(t), \forall t > 0$ .
- (ii)  $\|\hat{x}(t) - x(t)\| \rightarrow 0$  as  $t \rightarrow \infty$ .

- If (i) holds for any  $x(0), \hat{x}(0)$ , then the observer is **global**.
- If (ii) holds with exponential convergence, then the observer is **exponential**
- If (ii) holds with a tunable convergence rate, then the observer is **tunable**

From (1.1) and (1.3), an observer error is defined as:

$$\tilde{x} = x - \hat{x} \quad (1.4)$$

The prerequisites needed for any eventual solutions to the auxiliary system (1.3) will be illustrated in the sequel.

### 1.2.2 Observability

This subsection is devoted to the discussion of various conditions which are required to design the observer (1.3) for a given system (1.1). Such conditions are known as **observability conditions**. According to [29], these conditions are classified into two wide classes

1. Geometric conditions - conditions based on initial states ( $x_0$ )
2. Analytic conditions - conditions based on input  $u(t)$

#### 1.2.2.1 Geometric conditions

The designed observer must be able to extract the state information from the initial time via measured output. And more particularly, it must identify the initial values of the states. It means that the designed observer must be able to differentiate between various initial states, or analogously, it cannot hold indistinguishable states, which is defined as follows:

##### Definition 1.2: Indistinguishability [29]:

- A pair  $(x_0, x_0^1) \in \mathbb{R}^n \times \mathbb{R}^n$  is indistinguishable for a system (1.1) if

$$\forall u \in \mathcal{U}, \quad \forall t \geq 0, \quad h(\Phi_0(t, x_0)) = h(\Phi_0(t, x_0^1)) \quad (1.5)$$

- A state  $x$  is **indistinguishable** from  $x_0$  if the pair  $(x, x_0)$  is indistinguishable.

From Definition 2, the observability is defined as:

##### Definition 1.3: Observability [29] :

A system (1.1) is **observable** if it does not admit any indistinguishable pair (any state indistinguishable from  $x_0$ )

The definition of observability is quite generalised (more global). However, in several cases, the indistinguishability of states can be achieved in the neighbourhood but not globally.

For instance, consider the following system:

$$\dot{x} = u; \quad y = \sin x \quad (1.6)$$

It is easy to show that system contains indistinguishable states between  $]-\frac{\pi}{2}, \frac{\pi}{2}[$  but not in  $[0, 2\pi]$ . It implies that this system is not observable in  $[0, 2\pi]$  but observable in  $]-\frac{\pi}{2}, \frac{\pi}{2}[$ . It leads to the introduction of the notion of weak observability and local weak observability.

**Definition 1.4: Weak observability [29]:**

A system (1.1) is **weakly observable** if there is no indistinguishable state from  $x$  in a neighbourhood  $U$  for any  $x$ .

Weak observability is a weak notion of observability as compared to Definition 3.

**Definition 1.5: Local weak observability [29]:**

A system (1.1) is **locally weakly observable** if there exists a neighbourhood  $U$  of any  $x$  such that for any neighbourhood  $V$  contained in  $U$ , there is no indistinguishable state from  $x$  in  $V$  when considering time intervals for which trajectories remain in  $V$ .

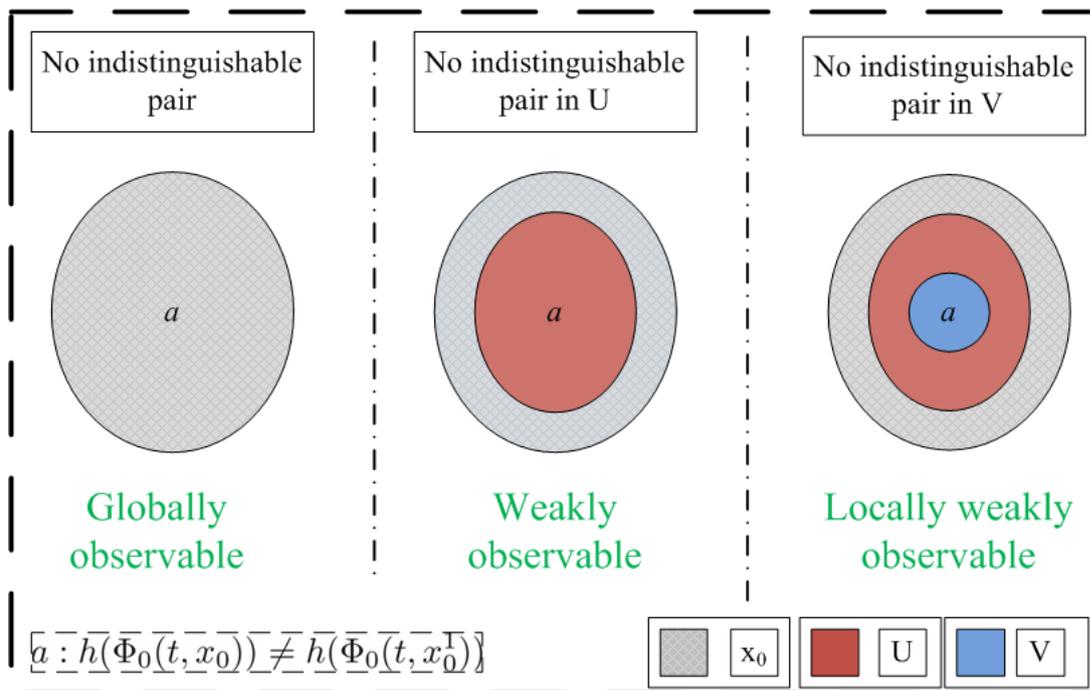


Figure 1.4: Schematic representation of geometric conditions of observability

The concept of local weak observability is more local than weak observability. The schematic representation of the geometric conditions is illustrated in Figure 1.4.

The notion of a weakly observable system implies that there exists a particular space where the states of systems are observable. It leads to the following definition:

**Definition 1.6: Observation Space [29]:**

The **observation space** for a system (1.1) is defined as the smallest real vector ( $\mathcal{O}(h)$ ) of  $\mathcal{C}^\infty$  functions containing the components of  $h$  and closed under Lie derivative along  $f_u := f(\cdot, u)$  for any constant  $u \in \mathbb{R}^s$ , i.e.,

$$\text{for any } \psi \in \mathcal{O}(h), L_{f_u}(\psi(x)) \in \mathcal{O}(h), \text{ where } L_{f_u}(\psi(x)) = \frac{\partial \psi}{\partial x}(f(x, u)).$$

The observation space describes the observability region and it characterizes the notion of rank condition.

**Definition 1.7: Observability rank condition [29] (resp. at  $x_0$ ):**

A system is said to satisfy the **observability rank condition** (resp. at  $x_0$ ) if

$$\forall x, \quad \dim(d\mathcal{O}(h)|_x) = n \quad [\text{resp. } \dim(d\mathcal{O}(h)|_{x_0}) = n], \quad (1.7)$$

where  $d\mathcal{O}(h)|_x$  is a set of  $d\psi(x)$  with  $\psi \in \mathcal{O}(h)$

From Definition 7, the authors of [30] stated the following lemma:

**Lemma 1.1: [30]**

A system (1.1) satisfying the observability rank condition (1.7) at  $x_0$  is **locally weakly observable** at  $x_0$ .

If  $\dim(d\mathcal{O}(h)|_x) = n$ , then it means that there exist  $n$  elements of the observation space which define a diffeomorphism around  $x_0$ . The observability condition is explained through the following examples:

**Example 1.1**

Let us consider a linear system as follows:

$$\dot{x} = Ax, \quad y = Cx, \quad x \in \mathbb{R}^n. \quad (1.8)$$

The observability rank condition of the system (1.8) is equivalent to  $\text{rank}(\mathcal{O}_m) = n$  with observability matrix  $\mathcal{O}_m$  which is given by

$$\mathcal{O}_m = \begin{bmatrix} L_f^1 y \\ L_f^2 y \\ \vdots \\ L_f^n y \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix},$$

where  $L_f^i y$  represents  $i^{\text{th}}$  Lie-derivative of  $y$ .

The pair  $(A, C)$  is called **observable** if system (1.8) fulfils (1.7).

**Example 1.2**

For the system (1.6), the observability matrix is given by

$$\mathcal{O}_m = \begin{bmatrix} L_f^1 h(x) \\ L_f^2 h(x) \end{bmatrix} = \begin{bmatrix} \cos x \\ -\sin x \end{bmatrix},$$

It is clearly seen that  $\dim(d\mathcal{O}(h)|_{x_0}) = 1$  for any  $x_0$ . Hence, system (1.6) is **observable**.

Similarly, for the system (1.1), the observability matrix is given by,

$$\mathcal{O}_m = \begin{bmatrix} L_f^1 h(x) \\ L_f^2 h(x) \\ L_f^3 h(x) \\ \vdots \\ L_f^n h(x) \end{bmatrix}.$$

From Definition 7 and Lemma 1.1, the system (1.1) is locally weak observable if  $\dim(d\mathcal{O}(h)|_{x_0}) = n$  for any  $x_0$ . More details about the observability rank condition of the nonlinear systems can be found in [31] and [29].

One can easily develop the observer for the system (1.1) if it satisfies the observability rank condition. However, these conditions are not sufficient to design an observer for the system (1.1). In the subsequent part, the remaining conditions are discussed.

### 1.2.2.2 Analytic conditions

The geometric observability conditions rely on the initial states of the systems. Whereas, in some systems, the observability is dependent on the input of the system also. This can be exemplified by the following example:

$$\dot{x} = \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} x; \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x. \quad (1.9)$$

The system (1.9) is observable for any constant input  $u$  except at  $u = 0$ . It means to design the observer, there is a need for certain conditions on the inputs to achieve observability. These types of conditions are known as **analytic conditions**.

To ensure an adequate comprehension of analytic properties, the following functions are defined.

#### Definition 1.8: Class $\mathcal{K}$ functions and $\mathcal{KL}$ function [32]

1. A continuous function  $\alpha : [0, a) \rightarrow [0, \infty)$  is said to belong to class  $\mathcal{K}$  if it is strictly increasing and  $\alpha(0) = 0$ .
2. It is said to belong to class  $\mathcal{K}_\infty$  if  $a = \infty$ , and  $\alpha(r) \rightarrow \infty$  as  $r \rightarrow \infty$ .
3. A continuous function  $\beta : [0, a) \rightarrow [0, \infty)$  is said to belong to class  $\mathcal{KL}$  if, for each fixed  $s$ , the mapping  $\beta(r, s)$  belong to class  $\mathcal{K}$  with respect to  $r$ , and for each fixed  $r$ , the

mapping  $\beta(r, s)$  is decreasing with respect to  $s$  and  $\beta(r, s) \rightarrow 0$  as  $s \rightarrow \infty$ .

Further, the definitions of universal inputs and uniform observability are provided.

**Definition 1.9: Universal input (resp. on  $[0, t]$ ) [29]**

- An input  $u$  is **universal** for (1.1) if

$$\forall x_0 \neq x_0^1, \exists \tau \geq 0 \text{ such that } h(\Phi_u(\tau, x_0)) \neq h(\Phi_u(\tau, x_0^1)). \quad (1.10)$$

- The condition (1.10) is equivalent to:

$$\int_t^t \|h(\Phi_u(\tau, x_t)) - h(\Phi_u(\tau, x_t^1))\|^2 d\tau > 0, \forall x_0 \neq x_0^1 \quad (1.11)$$

An input  $u$  is said to be **singular** if it is not universal. For instance, in the system (1.9),  $u(t) = 0$  is a singular input.

**Definition 1.10: Uniformly observable system [29]**

The system (1.1) is **uniformly observable** if every input of the system is universal (resp. on  $[0, t]$ ).

The aforementioned property implies that the observability of the systems is independent of the inputs. However, there is a need for some conditions to ensure the universality of the input over a particular time interval. In the sequel, these conditions are illustrated.

**Definition 1.11: Persistent input [29]**

An input  $u$  is **persistent input** of the system (1.1) if

$$\exists t_0, T : \forall t \geq t_0, \forall x_t \neq x_t^1, \int_t^{t+T} \|h(\Phi_u(\tau, x_t)) - h(\Phi_u(\tau, x_t^1))\|^2 d\tau > 0 \quad (1.12)$$

Though this condition preserves the observability over a given time interval, it does not eliminate the possibility that observability may disappear when  $t \rightarrow \infty$ . Thus, it leads to the following definition:

**Definition 1.12: Regularly persistent input [29]**

An input  $u$  is **regularly persistent input** of the system (1.1) if

$$\exists t_0, T : \forall t \geq t_0, \forall x_{t-T} \neq x_{t-T}^1, \forall t \geq t_0, \int_{t-T}^t \|h(\Phi_u(\tau, x_{t-T})) - h(\Phi_u(\tau, x_{t-T}^1))\|^2 d\tau \geq \beta(\|x_{t-T} - x_{t-T}^1\|), \quad (1.13)$$

for some function  $\beta \in \text{class } \mathcal{K}$ .

This condition preserves the observability when  $t \rightarrow \infty$  and it is very useful in the effective state estimation. It relies on the amount of time  $T$  needed to gather sufficient information. However, if one wants to estimate the states over a short interval of time, then this property is quite restrictive. For this purpose, the following property is defined.

**Definition 1.13: Locally regular input [29]**

An input  $u$  is locally regular input if

$$\begin{aligned} \exists t_0, T : \forall t \geq t_0, \forall x_{t-T} \neq x_{t-T}^1, \forall t \geq t_0, \\ \int_{t-T}^t \|h(\Phi_u(\tau, x_{t-T}) - h(\Phi_u(\tau, x_{t-T}^1))\|^2 d\tau \geq \beta(\|x_{t-T} - x_{t-T}^1\|, \frac{1}{T}), \end{aligned} \quad (1.14)$$

for some functions  $\beta \in \text{class } \mathcal{KL}$ .

The positive outcomes of each analytical property are depicted in Table 1.1.

Table 1.1: Summary of the analytic conditions of observability

Input characteristic	Advantages
Uniform input	System is uniformly observable
Persistent input	System is observable over a given interval of the time
Regular persistent input	System is observable at $t = \infty$ .
Locally regular input	Useful for the estimation over shorter duration

**Remark 1.1**

If considered system (1.1) is not observable, but it holds:

$$\begin{aligned} \forall u \text{ such that } x_0 \text{ and } x_0^1 \text{ are indistinguishable by } u : \\ \Phi_u(t, x_0) - \Phi_u(t, x_0^1) \rightarrow 0 \text{ as } t \rightarrow \infty. \end{aligned}$$

Then, system (1.1) satisfies the property of **detectability**.

In that case, one can propose the observer in the sense of (i) and (ii).

The above-mentioned conditions are sufficient to design the observer for a given system. However, an observer must estimate the states of the systems accurately, i.e., the estimation error of the observer (1.4) should zero (or  $\tilde{x} \rightarrow 0$ ) at  $t \rightarrow \infty$ . In the sequel, the stability conditions will be recalled from the literature.

### 1.2.3 Stability

This subsection is dedicated to an overview of stability properties. For simplicity of representation, the following part focuses on the stability of the states ( $x$ ) of the system (1.1). Whereas, in the observer design problems, the stability of the error dynamic (i.e.,  $\dot{\tilde{x}}$ , where  $\tilde{x} = x - \hat{x}$ ) is

essential. It is easy to deploy the stability conditions on  $\dot{\hat{x}}$  if one understands how to implement these conditions on the system (1.1).

For better comprehension, let us examine the stability of an autonomous system which is described as follows:

$$\dot{x} = f(x), \quad (1.15)$$

where  $x \in \mathcal{D}$  and  $f : \mathcal{D} \rightarrow \mathbb{R}^n$  is a locally Lipschitz function and  $\mathcal{D} \subset \mathbb{R}^n$ . Let  $x_e = 0$  be the equilibrium point of (1.15).

**Definition 1.14: Stability [33]**

The equilibrium point  $x_e = 0$  of the system (1.1) is

- **stable** if, for each  $\epsilon > 0$ , there is  $\delta = \delta(\epsilon, 0) > 0$  such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq 0. \quad (1.16)$$

- **unstable** if it is not stable.
- **asymptotically stable** if it is stable and  $\delta$  can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t \rightarrow \infty} x(t) = 0. \quad (1.17)$$

Definition 14 is local as the stability is restricted to the equilibrium point  $x_e = 0$ . The global stability of the system can be achieved if it is stable for all initial conditions  $x_e \in \mathbb{R}^n$ . In practical situations, the global stability of the system is always preferred. The given system (1.15) is said to be globally stable if it holds (1.16) for all initial conditions  $x_e \in \mathbb{R}^n$ . It is easy to notice that the autonomous system (i.e., (1.15)) fulfils the conditions of stability and asymptotic stability uniformly at the equilibrium point  $x_e = 0$  w.r.t. the initial time ( $t_0 = 0$ ).

Now, let us define the stability condition for the non-autonomous system (1.1). For this system, it needs to be recalled that  $t_0 = 0$  is the initial time of system (1.1).

**Definition 1.15: Stability [33]**

The equilibrium point  $x_e = 0$  of the system (1.15) is

- **stable** if, for each  $\epsilon > 0$ , there is  $\delta = \delta(\epsilon, 0) > 0$  such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq 0. \quad (1.18)$$

- **unstable** if it is not stable.
- **asymptotically stable** if it is stable and there exist a positive constant  $\rho = \rho(0)$  such that

$$x(t) \rightarrow 0 \text{ as } t \rightarrow \infty, \quad \forall \|x(0)\| < \rho. \quad (1.19)$$

- **uniformly stable** if and only if, it holds

$$\|x(t)\| \leq \alpha(\|x_0\|), \quad \forall t \geq 0, \quad \forall \|x_0\| \leq a, \quad (1.20)$$

where  $\alpha$  is a class  $\mathcal{K}$  function,  $x_0$  indicates an initial value of  $x$  (i.e.,  $x(0)$ ), and  $a$  is a positive scalar, independent of  $t_0 = 0$ .

- **uniformly asymptotically stable** if and only if it satisfies

$$\|x(t)\| \leq \beta(\|x_0\|), \quad \forall t \geq 0, \quad \forall \|x_0\| \leq a, \quad (1.21)$$

where  $\beta$  is a class  $\mathcal{KL}$  function, and  $a > 0$  is a constant, independent of  $t_0 = 0$ .

- **globally asymptotically stable** if and only if it holds (1.21) for any initial state  $x_0$ .

If the class  $\mathcal{KL}$  function  $\beta$  is considered in the form of an exponential function in (1.21), then a special case of uniformly asymptotic stability is obtained. Let us define this condition as follows:

#### Definition 1.16: Exponential stability

The equilibrium point  $x_e = 0$  of the system (1.1) is **exponentially stable** if there exist positive scalars  $a$ ,  $b$ , and  $c$  such that

$$\|x(t)\| < b\|x_0\|e^{-ct}, \quad \forall \|x_0\| < a \quad (1.22)$$

The system (1.1) is **globally exponentially stable** if it holds (1.22) for any initial state  $x_0$ .

After defining the stability and asymptotic stability of a system at an equilibrium point, the next objective is to illustrate methods which aid to determine the stability. The subsequent part is focused on such methods.

### 1.2.4 Unveiling Lyapunov theory: a comprehensive study of system stability

The necessary condition for the system (1.1) to be stable at  $x_e = 0$  is described by (1.16). However, condition (1.16) needs an explicit solution of the differential equation (1.1), which is often very difficult to compute or impossible to calculate in some cases. Thus, in the control domain, the well-known **Lyapunov theory** is widely used to investigate the stability of systems. The Lyapunov functions are extensively used for the analysis of the stability of the system without solving it explicitly. In this method, the energy of the system is represented in terms of a function. Further, the rate of change of the energy function is exploited to ascertain the stability of the system.

#### Definition 1.17: Lyapunov functions and Lyapunov surface [33]

1. Let  $\mathbb{D} \subset \mathbb{R}^n$  be a domain containing  $x_e = 0$ . If there exists a continuously differentiable function  $V : \mathbb{D} \rightarrow \mathbb{R}$  such that it holds:

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } \mathbb{D} - 0, \quad (1.23)$$

$$\dot{V}(x) \leq 0 \text{ in } \mathbb{D}, \quad (1.24)$$

where  $\dot{V}(x) = \frac{\partial V(x)}{\partial x} \dot{x} = \frac{\partial V(x)}{\partial x} f(x(t), u(t))$ . Then,  $V(x)$  is called as a **Lyapunov function**.

2. The surface  $V(x) = c$ , for some  $c > 0$ , is called as a **Lyapunov surface**.

Let us introduce a few more definitions which will be used to define the Lyapunov stability theorem.

**Definition 1.18: Definiteness of function [33]**

1. If a function  $V(x)$  holds conditions (1.23) and (1.24), i.e.,  $V(0) = 0$  and  $V(x) > 0$  for  $x \neq 0$ , then, it is said to be a **positive definite function**.
2. If  $V(x)$  fulfils weaker condition  $V(x) \geq 0$  for  $x \neq 0$ , then, it is said to be a **positive semidefinite function**.
3. However,  $V(x)$  is said to be a **negative definite** or **negative semidefinite** if  $-V(x)$  is a positive definite function or a positive semidefinite function, respectively.

The conditions  $V(0) = 0$  and  $V(x) > 0$ , for  $x \neq 0$  imply that the solution trajectory of  $\dot{V}(x)$  lies inside a ball  $\delta_0 \subset \mathbb{D}$ . The condition  $\dot{V}(x) \leq 0$  indicates that when a trajectory crosses a Lyapunov surface  $V(x) = c, c > 0$ , it enters inside the set  $\mathcal{V}_c = \{x \in \mathbb{R}^n \mid V(x) \leq c\}$  and can never come out again. Whereas in the case of  $\dot{V}(x) < 0$ , the trajectories move to the next Lyapunov surface  $V(x) = c_1, c > c_1 > 0$  from the surface  $V(x) = c$ . The Lyapunov surface  $V(x) = c$  shrinks to the origin as  $c$  decreases. Hence, the trajectory approaches the origin.

By using Definitions 17 and 18, the authors of [33] proposed the following theorem.

**Theorem 1.1**

The system (1.1) is **stable** at equilibrium point  $x_e = 0$  if it admits a continuously differentiable positive definite function  $V(x)$  so that  $\dot{V}(x)$  is negative semidefinite, i.e.,

$$\dot{V}(x) \leq 0 \text{ in } \mathbb{D}.$$

Further, the system (1.1) is **asymptotically stable** if it holds a continuously differentiable positive definite function  $V(x)$  so that  $\dot{V}(x)$  is negative definite, i.e.,

$$\dot{V}(x) < 0 \text{ in } \mathbb{D} - \{0\}.$$

Theorem 1.1 provides the sufficient condition for global stability of the system (1.1) at equilibrium point  $x_e = 0$  in the sense of Lyapunov. While the system (1.1) is said to be locally stable at an equilibrium point  $x_e = 0$  if it admits a Lyapunov function  $\forall x \in \mathcal{B}$ , where  $\mathcal{B} \subseteq \mathbb{D} \subset \mathbb{R}^n$  is a particular domain.

In addition to this, there is no systematic approach proposed in the literature to determine the Lyapunov function for a particular system. For example, in electrical or mechanical systems, we can consider their energy function as a Lyapunov function candidate. In other cases, it is primarily an instance of trial and error.

The Lyapunov method can be easily illustrated through the following example:

**Example 1.3: [33]**

Let us consider the following sets of equations which represent the pendulum model without

friction:

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= -a \sin x_1,\end{aligned}\tag{1.25}$$

where  $x_1$  and  $x_2$  denote the angle subtended by the rod of the pendulum, and the angular velocity of the pendulum, respectively.  $a$  is a scalar. For checking the stability of system (1.25) at the origin, a Lyapunov function is defined as:

$$V(x) = a(1 - \cos x_1) + \frac{1}{2}x_2^2.$$

It is also known as the energy function of the system (1.25). Further, the derivative of  $V(x)$  is computed as follows:

$$\dot{V}(x) = a \sin x_1 \dot{x}_1 + \frac{1}{2}2x_2 \dot{x}_2 = ax_2 \sin x_1 - ax_2 \sin x_1 = 0.$$

Clearly,  $V(0) = 0$ ,  $V(x) > 0 \forall x_1 \in ]-2\pi, 2\pi[$ , and  $\dot{V}(x) \equiv 0$ . Therefore,  $V(x)$  fulfils (1.23) and (1.24). Hence, system (1.25) is stable at the origin as it admits a Lyapunov function. However, since  $\dot{V}(x) \equiv 0$ , it is not asymptotically stable.

The aforementioned example demonstrates how the Lyapunov function is utilised in stability analysis. Furthermore, the sufficient condition to achieve the exponential stability of the system at its equilibrium point is stated as follows:

**Theorem 1.2:** [33]

- The system (1.1) is **exponentially stable** at an equilibrium point  $x_e = 0$  if it admits a continuously differentiable function  $V : [0, \infty) \times \mathbb{D} \rightarrow \mathbb{R}$  such that:

$$a_1 \|x\|^b \leq V(t, x) \leq a_2 \|x\|^b,\tag{1.26}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -a_3 \|x\|^b,\tag{1.27}$$

$\forall t \geq 0$ ,  $\forall x \in \mathbb{D}$ , where  $a_1, a_2, a_3$  and  $b$  are positive scalars, and  $\mathbb{D} \subset \mathbb{R}^n$  is a domain containing  $x_e = 0$ .

- If the assumptions hold globally, then the system is **globally exponentially stable**.

The following example demonstrates the comprehension of Theorem 1.2 .

**Example 1.4:** [33]

Let us consider

$$\begin{aligned}\dot{x}_1 &= -x_1 - g(t)x_2, \\ \dot{x}_2 &= x_1 - x_2,\end{aligned}\tag{1.28}$$

where  $g(t)$  is continuously differentiable and fulfils:

$$0 \leq g(t) \leq k, \text{ and } \dot{g}(t) \leq g(t), \forall t \geq 0, \quad (1.29)$$

where  $k$  is a positive scalar.

For analysing the stability of (1.28), the Lyapunov function is considered as:

$$V(t, x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\top \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 + g(t) \end{bmatrix}}_P \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

It is easy to interpret that  $V(x)$  holds:

$$x_1^2 + x_2^2 \leq V(x) \leq x_1^2 + (1+k)x_2^2, \forall x \in \mathbb{R}^2. \quad (1.30)$$

Hence,  $V(x)$  satisfies (1.26) with  $a_1 = 1$ ,  $a_2 = 1 + a$  and  $b = 2$ .

Now,  $\dot{V}(x)$  is given by

$$\dot{V} = -2x_1^2 + 2x_1x_2 - (2 + 2g(t) - \dot{g}(t))x_2^2.$$

Since  $\dot{g}(t) \leq g(t)$ ,  $(2 + 2g(t) - \dot{g}(t)) \geq 2 + 2g(t) - g(t) \geq 2$ ,

$$\dot{V} \leq -2x_1^2 + 2x_1x_2 - 2x_2^2 \leq - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\top \underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_Q \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

As  $Q > 0$ ,  $\dot{V}(x)$  holds (1.27) with  $a_3 = \lambda_{\max}(Q) = 3$ . Thus, from Theorem 1.2, system (1.28) is globally exponentially stable at  $x_e = 0$ .

In the case of the system (1.1), the stability of the system also depends on the input. In the sequel, this input-based stability of the system is discussed.

### 1.2.5 Reiterating ISS notion: grasping its Impact on the system stability

If we consider the system (1.1) as the unforced system, i.e.,  $u(t) = 0$ , then it is globally uniformly asymptotically stable at the equilibrium point  $x_e = 0$ . But what if we consider a class of nonlinear systems:

$$\dot{\zeta} = f(\zeta, u, t), \quad (1.31)$$

where  $f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is piece-wise continuous in  $t$ , and it is assumed to be locally Lipschitz in  $x$  and  $u$ . The input of the system  $u(t)$  is considered as a piece-wise continuous bounded function of  $t$ .

In order to establish the stability condition of the system (1.31), the concept of ISS is used. In [34], an ISS notion was introduced in the control system domain. Further, the authors of [35] proposed an ISS-Lyapunov function to use the ISS property for the stability of systems. Let us recall the definitions of the ISS notion and an ISS-Lyapunov function.

**Definition 1.19: ISS property [34]**

If there exists a class  $\mathcal{KL}$  function  $\beta$  and a class  $\mathcal{K}$  function  $\gamma$  such that for any initial state  $\zeta(0)$ , and any bounded input  $u(t)$ , solution  $\zeta(t)$  exists for all  $t \geq 0$  and satisfies:

$$\|\zeta(t)\| \leq \beta(\|\zeta(0)\|, t) + \gamma(\|u\|_\infty), \quad \forall t \geq 0. \quad (1.32)$$

Then, the system (1.31) is **input-to-state stable**.

The notion of ISS is often used in the literature (for instance, [36], [37]) for examining the robustness of nonlinear systems with regard to their inputs across a wide variety of applications. It is a challenging task to demonstrate the ISS of a system. Thus, the following ISS-Lyapunov function is widely used for the ISS analysis of the systems.

**Definition 1.20: ISS-Lyapunov function [35]**

- A smooth function  $V(\zeta) : \mathbb{R}^n \rightarrow \mathbb{R}$  is called as **an ISS-Lyapunov function** for the system (1.31) if there exist  $\mathcal{K}_\infty$  functions  $\alpha_1, \alpha_2$ , and  $\mathcal{K}$  functions  $\chi_1, \chi_2$ , such that

$$\alpha_1(\|\zeta\|) \leq V(\zeta) \leq \alpha_2(\|\zeta\|) \quad (1.33)$$

$$\dot{V}(\zeta, u) \leq -\chi_1(\|\zeta\|), \quad (1.34)$$

for any  $\zeta \in \mathbb{R}^n$  and  $u \in \mathbb{R}^s$  so that  $\|\zeta\| \geq \chi_2(\|u\|)$ .

- Further, the authors of [35] have provided another definition of the ISS-Lyapunov function which is as follows: A smooth function  $V(\zeta) : \mathbb{R}^n \rightarrow \mathbb{R}$  is called an ISS-Lyapunov function for the system (1.31) if and only if there exist  $\alpha_i \in \mathcal{K}_\infty$  ( $1 \leq i \leq 4$ ) functions such that (1.33) holds along with

$$\dot{V}(\zeta, u) \leq -\alpha_3(\|\zeta\|) + \alpha_4(\|u\|). \quad (1.35)$$

Clearly, (1.35) implies (1.34).

Further, the authors of [35] stated the following theorem:

**Theorem 1.3**

The system (1.31) is **ISS** w.r.t. the input  $u(t)$  if and only if it admits an ISS-Lyapunov function.

One can use example 4 to illustrate Theorem 1.3 .

The next section of the chapter is dedicated to the summary of a few observer design methodologies.

### 1.3 State-of-the-art on nonlinear observer design

State estimation techniques for linear systems have been extensively studied and proven to be quite reliable. The Kalman filter [38] and the Luenberger observer [39] were the first state estimation approaches designed for linear systems. In contrast to linear systems, designing observers for nonlinear systems remains a difficult task. As a result, a substantial amount of

research is conducted in this domain, and several methods for each type of system are developed. All these techniques have their own advantages and limitations. Several variants of nonlinear observer approaches are described in Figure 1.5.

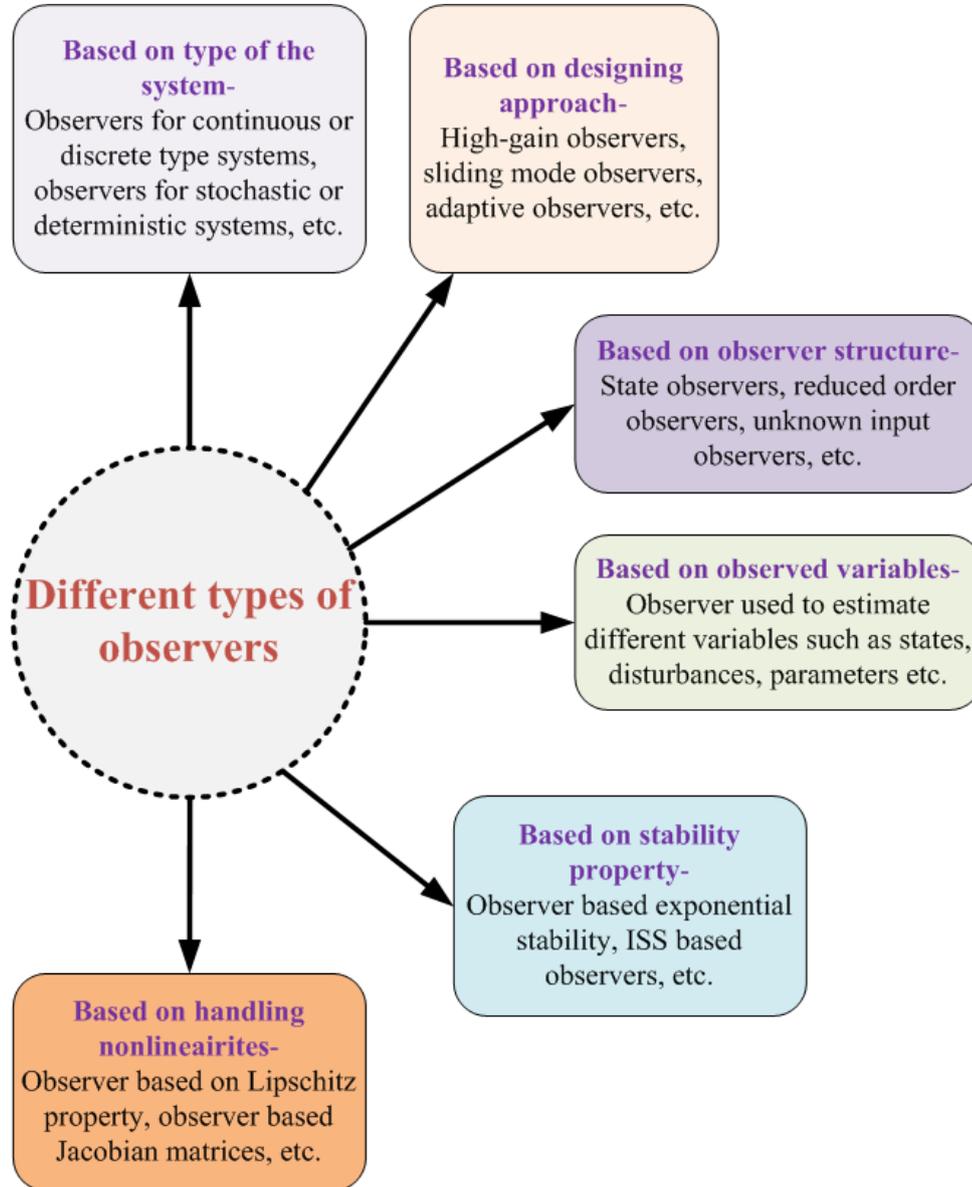


Figure 1.5: Various observer design methods

The subsequent subsection outlines a short recap of the various observer design methodologies.

### 1.3.1 Nonlinear transformation based method

The authors of [40] introduced the methodology based on coordinate transformation for designing the observers for the nonlinear autonomous system.

Consider a class of nonlinear autonomous systems in the subsequent form:

$$\begin{aligned}\dot{x} &= f(x), \\ y &= g(x),\end{aligned}\tag{1.36}$$

where  $x \in \mathbb{R}^n$ ,  $y \in \mathbb{R}^p$  are state vectors and output measurements, respectively.  $f(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $g(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^p$  are the nonlinearities of the system dynamics and measurements, respectively. In this method, by using a **nonlinear change of coordinates**  $Z = \phi(x)$ , the nonlinear system (1.36) is transformed into the linear system under the following form:

$$\begin{aligned}\dot{z} &= A_z z + f_z(y), \\ y &= C_z z.\end{aligned}\tag{1.37}$$

The matrices  $A_z$  and  $C_z$  are represented under the Brunovsky dual form [41], i.e.,

$$A_z = \begin{bmatrix} \mathbb{O}_{n-1} & \mathbb{I}_{n-1} \\ \mathbb{O} & \mathbb{O}_{n-1}^\top \end{bmatrix}; \quad C_z = [\mathbb{I}_1 \quad \mathbb{O}_{n-1}^\top].$$

The form of the transformed linear system (1.37) is known as the **canonical observer form**. Further, the Luenberger observer for the system (1.37) is considered for the estimation purpose which is given by

$$\dot{\hat{z}} = A_z \hat{z} + f_z(y) + L(y - C_z \hat{z}),\tag{1.38}$$

where  $\hat{z}$  is an estimated state, and  $L$  is an observer gain needs to be determined. The estimation error of the observer is defined as  $\tilde{x} = x - \hat{x}$ .

From (1.37) and (1.38), we obtain:

$$\dot{\tilde{x}} = (A_z - LC_z)\tilde{x}.\tag{1.39}$$

The gain  $L$  is computed by using pole placement conditions.

Furthermore, in [42], this method is extended for the systems with multiple outputs.

The authors of [43] considered the ensuing system for the design purpose:

$$\begin{aligned}\dot{x} &= f(x, u), \\ y &= g(x, u),\end{aligned}\tag{1.40}$$

where  $u \in \mathbb{R}^s$  is the input of the systems. Further, the following nonlinear coordinate transformation is employed for the observer development:

$$\begin{aligned}z &= \phi(x, u_1), \\ y_z &= \psi(x, u_1),\end{aligned}\tag{1.41}$$

where  $u_1 = [u \quad u_d^1 \quad \dots \quad u_d^n]^\top$ , and  $u_d^i$  represents  $i^{\text{th}}$  derivative of  $u$ . Under this transformation, the system (1.40) is reformulated as

$$\begin{aligned}\dot{z} &= A_z z + f_z(y, u_1), \\ y_z &= C_z z.\end{aligned}\tag{1.42}$$

The subsequent observer is considered under the assumption that all the terms  $u_d^i$  exist:

$$\begin{aligned}\dot{\hat{z}} &= A_z \hat{z} + f_z(y, u_1) + L(y_z - \hat{y}_z), \\ \hat{y}_z &= C_z \hat{z}.\end{aligned}\tag{1.43}$$

The dynamic of the estimation error of the observer (1.43) is identical to the one described in (1.39). Observer gain is computed by using pole placement methods. Hence, the observer based on the nonlinear transformation can be implemented for multi-input systems.

In addition to this, several approaches based on the coordinate transformations are illustrated in [44], [45] [46] and [47]. The drawbacks of this approach are listed as follows:

1. The types of systems that exhibit the aforementioned coordinate transformations tend to be restricted and fail to encompass all classes of systems.
2. The method deployed for identifying such transformation is intricate and demands an elevated level of complexity.
3. For systems with multiple outputs, the transformed systems encompass all the derivatives of the input terms. It is quite restrictive in many scenarios, as derivatives of the input terms may not exist.

### 1.3.2 High-gain observers

Over the last few decades, triangular forms of systems have become a centre of research in control systems. The authors of [48] related the triangular structure of the systems to uniformly observable systems, while the phase-variable form was introduced for differentially observable systems in [49]. The high-gain observer for the phase variable form was presented in [50], while [51] dealt with the design of high-gain observers for the triangular forms. High-gain observers are extensively used in control domains for state or parameter estimation and for achieving the desired control. Some of the results were outlined in [52], [53], [7], [54], [55] and [56].

Let us consider the subsequent class of nonlinear systems which is widely used for high-gain observer design:

$$\begin{aligned}\dot{x} &= Ax + Gf(x, u), \\ y &= Cx,\end{aligned}\tag{1.44}$$

where  $x \in \mathbb{R}^n$  denotes state vectors,  $y \in \mathbb{R}^p$  represents output measurements, and  $u \in \mathbb{R}^s$  is inputs of the system.  $G = \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix}^\top$ ,  $C = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$ , and state matrix  $A$  is defined by,

$$(A)_{i,j} = \begin{cases} 1 & \dots\dots\dots i = j + 1, \\ 0 & \dots\dots\dots i \neq j + 1. \end{cases}\tag{1.45}$$

$f(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is the nonlinear part of the system dynamics which is assumed to be globally Lipschitz w.r.t.  $x$  and  $u$ , i.e., it holds (A.1). In addition to this, the pair  $(A, C)$  is considered as **detectable**. Further, the high-gain observer structure for the system (1.44) is given by,

$$\dot{\hat{x}} = A\hat{x} + Gf(\hat{x}, u) + L(y - C\hat{x}),\tag{1.46}$$

where  $\hat{x} \in \mathbb{R}^n$  is an estimated state and  $L$  is the observer gain which needs to be computed. From (1.44) and (1.46), the estimation error dynamic is obtained as follows:

$$\dot{\tilde{x}} = (A - LC)\tilde{x} + G(f(x, u) - f(\hat{x}, u)).\tag{1.47}$$

The aim of the proposed high-gain observer methodology is to compute the gain matrix  $L$  such that estimation error  $\tilde{x}$  is stable, and  $\tilde{x} \rightarrow 0$  when  $t \rightarrow \infty$ . The error dynamics (1.47) is altered based on the Lipschitz condition (A.1), and it leads to:

$$\dot{\tilde{x}} = (A - LC)\tilde{x} + G\gamma_f\tilde{x}. \quad (1.48)$$

The estimation error depicts a time-dependent rise for the high Lipschitz constant  $\gamma_f$ , as seen in (1.48). Thus, the gain  $L$  needs to be large enough such that it compensates for the huge difference between actual states and estimated states. Hence, the proposed observer with computed gain matrix  $L$  is called as **High-gain observer**.

In high-gain methodology, the gain matrix  $L$  is written under the following form:

$$L = T(\theta)K, \quad (1.49)$$

where  $T(\theta) = \text{diag}(\theta \ \theta^2 \ \dots \ \theta^n)$ ,  $\theta > 1$ , and  $K \in \mathbb{R}^{n \times p}$ . For further information, one can read [7]. The following theorem delivers a sufficient condition for the asymptotic stability of the error dynamic (1.47).

**Theorem 1.4:** [7]

If there exist symmetric positive definite matrix  $P$ , matrix  $R$ ,  $\delta > 0$  and  $\theta \geq 1$  such that

$$A^\top P + PA - C^\top R - R^\top C + \delta \mathbb{I} < 0, \quad (1.50)$$

$$\theta > \theta_0 = \frac{2\gamma_f \lambda_{\max}(P)}{\delta}, \quad (1.51)$$

then estimation error  $\tilde{x}$  is asymptotically stable for  $K = P^{-1}R$ .

**Proof:** One can find additional information on the proof of the theorem in [7], [57] and [58].  $\square$

Table 1.2: Summary of HGO approaches

No	Method	Important consequences	References
1	Astolfi/Marconi high-gain observer	provides a trade-off between estimating accuracy and the rate of convergence	[59], [60]
2	Reduced-order high-gain observer	aids to minimize the computation complexity of the observer, and provides robustness to disturbances and uncertainties	[61]
3	Adaptive high-gain observer	estimate the states of system along with unknown parameters efficiently, and offers a trade-off between estimating accuracy and the rate of convergence	[62], [63]

According to [56] and [64], the key benefit of the high-gain methodology is that it always ensures the existence of an exponentially convergent observer due to the tuning of only one parameter which should be selected to be large enough.

Recently, an abundance of research on high-gain observers has been conducted. Among these proposed methods, a few approaches are summarised in Table 1.2.

Although authors of [33] and [65] have demonstrated the efficiency of high-gain observers in output feedback control problems, the major drawback of the method is the use of high-gains. The limitation of high-gain in this approach is resolved in [7].

### 1.3.3 Luenberger observers

The Luenberger observer, presented in [39], is the first state estimation technique designed for linear systems and has manifested consistent reliability. Whereas it is not an easy task to develop an observer for nonlinear systems. As a result, there has been a tremendous amount of research done on the topic of the development of Luenberger observers, and various techniques have been established.

Let us consider the ensuing system:

$$\begin{aligned}\dot{x} &= Ax + Bu + f(x), \\ y &= Cx,\end{aligned}\tag{1.52}$$

where  $x \in \mathbb{R}^n$ ,  $y \in \mathbb{R}^p$  and  $u \in \mathbb{R}^s$  are state vectors, output measurements, and inputs of the system, respectively. The nonlinear function  $f(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is assumed to be globally Lipschitz w.r.t.  $x$ , i.e., it satisfies (A.1).  $A$ ,  $B$  and  $C$  are known constant matrices, and the pair  $(A, C)$  is presumed to be detectable. Then, system (1.52) admits the following observer structure:

$$\dot{\hat{x}} = A\hat{x} + Bu + f(\hat{x}) + L(y - C\hat{x}).\tag{1.53}$$

From (1.52) and (1.53), the error dynamic is obtained as:

$$\dot{\tilde{x}} = (A - LC)\tilde{x} + (f(x) - f(\hat{x})).\tag{1.54}$$

The observer (1.53) is known as the **Luenberger observer**, and it is widely used in control systems for state estimation purposes, for example, [19], [66], [10], [11], [12], [13], and so on. There are various methodologies that have been developed in the literature to derive the conditions for the stability of the error dynamic (1.54). Among these techniques, a few are recalled in the sequel.

#### 1.3.3.1 LMI-based approach

Over the last few decades, LMI-based observers have gained a lot of attention from researchers. Due to the rigorous use of Lyapunov theory, Lipschitz property of nonlinear functions, and other mathematical tools such as Schur complement and Young inequality, various novel LMI conditions have been developed which are presented in [18], [66], [12], [13], and [21].

The following theorem provides an LMI condition for the stability of the error dynamic (1.54):

#### Theorem 1.5: [18]

The estimation error dynamic (1.54) is asymptotically stable if there exist a symmetric positive definite matrix  $P$  and a matrix  $R$  of the appropriate dimension so that the ensuing

LMI condition holds:

$$\begin{bmatrix} A^\top P + PA - R^\top C - C^\top R + \mathbb{I}_n & P \\ P & -\frac{1}{\gamma_f^2} \mathbb{I}_n \end{bmatrix} < 0. \quad (1.55)$$

Further, gain matrix  $L$  is given by  $L = P^{-1}R^\top$ .

For the proof of the theorem, one can delve into [18].

The authors of [11] proposed an improvement in LMI condition (1.55) to tackle the problem of nonlinear systems having larger Lipschitz constants. The developed LMI condition is stated in the subsequent theorem:

**Theorem 1.6:** [11]

The estimation error dynamic (1.54) is asymptotically stable if there exist matrices  $P = P^\top > 0$  and  $R$  of appropriate dimension, and a positive scalar  $\delta$  so that the following LMI condition holds:

$$\begin{bmatrix} A^\top P + PA - R^\top C - C^\top R + \delta \gamma_f^2 \mathbb{I}_n & P \\ P & -\delta \mathbb{I}_n \end{bmatrix} < 0. \quad (1.56)$$

Further, gain matrix  $L$  is computed as:  $L = P^{-1}R^\top$ .

For more details about this, one can refer [11]. Furthermore, in the same article, the authors presented a less conservative Lipschitz condition which is illustrated in (A.2). Based on (A.2), the consequent corollary is presented:

**Corollary 1.1: LMI based on (A.2)**

If there exist matrices  $P = P^\top > 0$  and  $R$  of appropriate dimension, and a positive scalar  $\delta$  so that the successive LMI condition holds:

$$\begin{bmatrix} A^\top P + PA - R^\top C - C^\top R + \delta \mathcal{T}^\top \mathcal{T} & P \\ P & -\delta \mathbb{I}_n \end{bmatrix} < 0. \quad (1.57)$$

Then, the estimation error dynamic (1.54) is asymptotically stable. The gain matrix  $L$  is given by  $L = P^{-1}R^\top$ .

In addition to this, further enhancements to the standard LMI approach were proposed by the authors of [13, 14, 22], and [15].

### 1.3.3.2 Riccati equation-based method

Authors of [67] have employed ARE approach for the nonlinear observer design. Later on, in [19], a novel observer design technique is proposed for nonlinear systems. Further, the authors of [11] proposed an ARE as an alternative for the LMI condition. Both LMIs (1.57) and (1.56) contain two unknown variables  $P$  and  $L$ . The deployment of this approach reduces the computation of unknown variables. A sufficient condition for the observer design is illustrated as follows:

**Theorem 1.7:** [11]

If there exist  $\delta > 0$  and  $\beta \in \mathbb{R}$  such that the following Riccati inequality has a solution as  $P = P^\top > 0$ :

$$A^\top P + PA + \delta \mathcal{T}^\top \mathcal{T} + \frac{1}{\delta} P^\top P - \beta^2 C^\top C < 0. \quad (1.58)$$

Then the error dynamic (1.54) is asymptotically stable. Further, gain matrix  $L$  is given by  $L = \frac{\beta^2}{2} P^{-1} R^\top$ .

In theorem 1, since  $P$  is the only unknown, the Riccati inequality (1.58) is relatively simpler to solve than solving both LMIs (1.57) and (1.56). The efficiency of this method is demonstrated in [11].

In the literature, various observer methodologies are proposed for the state estimation of the systems. However, in many practical applications, the dynamics and measurements of the system are very susceptible to being corrupted by external disturbances or noise. In such scenarios, the proposed observers must be able to reconstruct the system's state by attenuating the noise or disturbances. In the sequel, such observer design techniques are recalled.

### 1.3.4 Revealing the nonlinear observer design for the disturbance-affected systems

Designing observers to estimate the states of the systems in the presence of noise and disturbances is one of the most challenging tasks in the control system. Due to noisy measurements and external disturbances, it is difficult to obtain the states precisely. In such circumstances, it is essential to develop the observers that perform within specified performance thresholds. Thus, the observer design for systems states estimation under the presence of disturbances/noise has become a critical issue.

For example,

- An  $\mathcal{H}_\infty$  criterion-based nonlinear observer has been proposed in [68] for the state-of-charge (SoC) estimation of the lithium-ion battery model.
- The authors of [69] and [70] developed an observer for the control of glucose levels in type-2 diabetic patients.
- In [14], the nonlinear observer is designed for the slip angle estimation of an autonomous vehicle model with noise-corrupted outputs and dynamics.

A few design methodologies are discussed in the subsequent part.

#### 1.3.4.1 ISS based approach

In Subsection 1.2.5, the notion of ISS is illustrated. The ISS approach enables an easy and robust examination of the influence of system and measurement disturbances on estimate error. From the Definition 19, it is easy to interpret that the states of the system are bounded for the given bounded input. Similarly, in ISS-based observer design methods, the estimation error of the observer is considered as the state, and noise  $\omega$  is taken as input to the system. Further, a sufficient condition on the gain matrix  $L$  is determined such that the estimation error dynamic is ISS w.r.t.  $\omega$ .

Let us consider a generalised class of nonlinear systems under the presence of external disturbances or noise as follows:

$$\begin{aligned}\dot{x} &= Ax + Bu + f(x) + E\omega, \\ y &= Cx + D\omega,\end{aligned}\tag{1.59}$$

where  $\omega \in \mathbb{R}^q$  is the disturbance or noise present in both system dynamics and measurements.  $E \in \mathbb{R}^{n \times q}$  and  $D \in \mathbb{R}^{p \times q}$  are known constant matrices and other parameters are the same as described in (1.52). The system (1.59) holds the following assumptions:

1. The pair  $(A, C)$  is detectable.
2. The nonlinear function  $f(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is globally Lipschitz w.r.t.  $x$ , i.e., it fulfils (A.1).

For the state estimation purpose, the ensuing observer structure is considered:

$$\dot{\hat{x}} = A\hat{x} + Bu + f(\hat{x}) + L(y - C\hat{x}).\tag{1.60}$$

From (1.59) and (1.60), the error dynamic is obtained as:

$$\dot{\tilde{x}} = (A - LC)\tilde{x} + (f(x) - f(\hat{x})) + (E - LD)\omega.\tag{1.61}$$

The objective of the observer design is to compute an observer gain  $L$  such that the estimation error  $\tilde{x}$  is an ISS w.r.t.  $\omega$ . The subsequent theorem of [32] showcases a sufficient LMI condition such that system (1.61) is an ISS w.r.t. to  $\omega$ .

**Theorem 1.8:** [32]

If there exist matrices  $P = P^\top > 0$ ,  $R$  of appropriate dimensions and a positive scalar  $\alpha$  such that

$$\begin{bmatrix} A^\top P + PA - R^\top C - C^\top R + \alpha\gamma_f^2 \mathbb{I} & P \\ \star & -\frac{\alpha}{2} \mathbb{I} \end{bmatrix} < 0,\tag{1.62}$$

then, the system (1.61) is an ISS w.r.t. the disturbance  $\omega$ . The gain matrix is computed as  $L = P^{-1}R^\top$ .

**Proof:** For more details about the proof of theorem 1.8, one can refer to [32]. □

The efficiency of ISS-based nonlinear observer is highlighted in [71], [72] and [37].

#### 1.3.4.2 $\mathcal{H}_\infty$ observer

An alternative for ISS-based observers is  $\mathcal{H}_\infty$  observers. For the simplicity of representation, let us consider a generalised class of nonlinear systems:

$$\begin{aligned}\dot{x} &= Ax + Bu + Gf(Hx) + E\omega, \\ y &= Cx + D\omega,\end{aligned}\tag{1.63}$$

where all parameters and variables are illustrated in (1.59).  $f(Hx)$  denotes the state-dependent nonlinearity, where  $H \in \mathbb{R}^{n \times n}$  is a known constant matrix. In (1.59), the nonlinearity is considered in a generalised way whereas in (1.63), the nonlinearity in the form of  $f(Hx)$  allows to introduce a constraint which is often found in the multi-variable systems.

The observer for the system (1.63) is given by

$$\dot{\hat{x}} = A\hat{x} + Bu + Gf(H\hat{x}) + L(y - C\hat{x}). \quad (1.64)$$

From (1.63) and (1.64),

$$\dot{\tilde{x}} = (A - LC)\tilde{x} + G(f(Hx) - f(H\hat{x})) + (E - LD)\omega. \quad (1.65)$$

Furthermore, it is assumed that  $f(Hx)$  satisfies:

$$(f(Hx) - f(H\hat{x}))^\top \left( (f(x) - f(\hat{x})) - KH\tilde{x} \right) \leq 0, \quad (1.66)$$

where  $K$  is a symmetric positive definite matrix of appropriate dimension.

The aim is to calculate the observer gain  $L$  such that the estimation error dynamic (1.65) holds

$$\|\tilde{x}\|_{\mathcal{L}_2^n} \leq \sqrt{\mu\|\omega\|_{\mathcal{L}_2^q}^2 + \nu\|\tilde{x}_0\|^2}, \quad (1.67)$$

where  $\mu > 0$  indicates the disturbance attenuation level and  $\nu > 0$  is to be determined.

The authors of [73] proposed the ensuing theorem which provides a sufficient condition for asymptotic stability of the error dynamic (1.65).

**Theorem 1.9:** [73]

If there exist scalars  $\tau \geq 0$ ,  $\sigma$  and a matrix  $P = P^\top > 0$  of appropriate dimension such that the subsequent optimization problem is solvable:

$$\begin{aligned} & \min \mu^2 \\ & \text{Subject to } P \geq t\mathbb{I}, \tau \geq 0, \text{ and} \\ & \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ \star & -\tau\mathbb{I} & 0 \\ \star & \star & -\mu^2\mathbb{I} - \sigma D^\top D \end{bmatrix} < 0, \end{aligned} \quad (1.68)$$

where

$$M_{11} = A^\top P + PA - \sigma C^\top C + \mathbb{I}, \quad (1.69)$$

$$M_{12} = PG + \frac{1}{2}\tau H^\top K^\top, \quad (1.70)$$

$$M_{13} = PE - \sigma C^\top D. \quad (1.71)$$

Then the estimation error dynamic (1.65) fulfils  $\mathcal{H}_\infty$  criterion (1.67). The gain matrix  $L$  is calculated as:  $L = \frac{1}{2}P^{-1}C$ .

One can refer to [73] for the proof of Theorem 1.9 .

Though the standard nonlinear Luenberger observers are proven to be quite reliable, there are certain nonlinear estimation techniques were established in the literature. Among these, one of the popular methods is recapitulated in the sequel.

### 1.3.5 A generalised Arcak's observer

The authors of [74] proposed a new observer configuration for the state estimation of nonlinear systems. In this method, a standard Luenberger observer structure is modified with the addition of a gain inside the nonlinear part of the system dynamics.

The class of generalised nonlinear systems used for this method is given by,

$$\begin{aligned}\dot{x} &= Ax + Bu + Gf(Hx), \\ y &= Cx,\end{aligned}\tag{1.72}$$

where  $x \in \mathbb{R}^n$ ,  $y \in \mathbb{R}^p$  and  $u \in \mathbb{R}^s$  are state vectors, output measurements, and inputs of the system, respectively.  $A$ ,  $B$ ,  $G$ ,  $H$  and  $C$  are known constant matrices of appropriate dimensions, and it is assumed that pair  $(A, C)$  is detectable. The nonlinear function  $f(Hx)$  is considered as a  $m$ -dimensional vector where each entry is a function of a linear combination of states, i.e.,

$$f_i = f_i\left(\sum_{j=1}^n \mathcal{H}_{ij}x_j\right), \quad i = 1, \dots, m.\tag{1.73}$$

. In addition to this, each  $f_i$  is assumed to be non-decreasing, i.e., it fulfils:

$$0 \leq \frac{f_i(x_1) - f_i(x_2)}{x_1 - x_2}, \quad \forall x_1 \neq x_2 \in \mathbb{R}.\tag{1.74}$$

For ease of comprehension, let us define  $\theta = Hx$ .

The structure of the proposed observer is represented as follows:

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + Gf(\hat{\theta}) + L_1(y - C\hat{x}), \\ \hat{\theta} &= Hx - L_2(y - C\hat{x}),\end{aligned}\tag{1.75}$$

where  $\hat{x}$  represents the estimated state obtained from the observer, and  $L_1, L_2$  are observer gain matrices of appropriate dimension. The pictorial illustration of Arcak's observer is displayed in Figure 1.6.

From (1.72) and (1.75), the estimation error dynamic is achieved as follows:

$$\dot{\tilde{x}} = (A - LC)\tilde{x} + G(f(\theta) - f(\hat{\theta})).\tag{1.76}$$

A sufficient condition to ensure the stability of the error dynamic (1.76) is described in the succeeding theorem.

**Theorem 1.10:** [74]

If there exist a symmetric positive definite matrix  $P$ , a positive scalar  $\epsilon$  and a diagonal matrix  $\Lambda$  such that

$$\begin{bmatrix} (A - L_1C)^\top P + P(A - L_1C) + \epsilon\mathbb{I} & PG + (H - L_2C)^\top \Lambda \\ \star & 0 \end{bmatrix} < 0\tag{1.77}$$

Then, the estimation error dynamic (1.76) is exponentially stable.

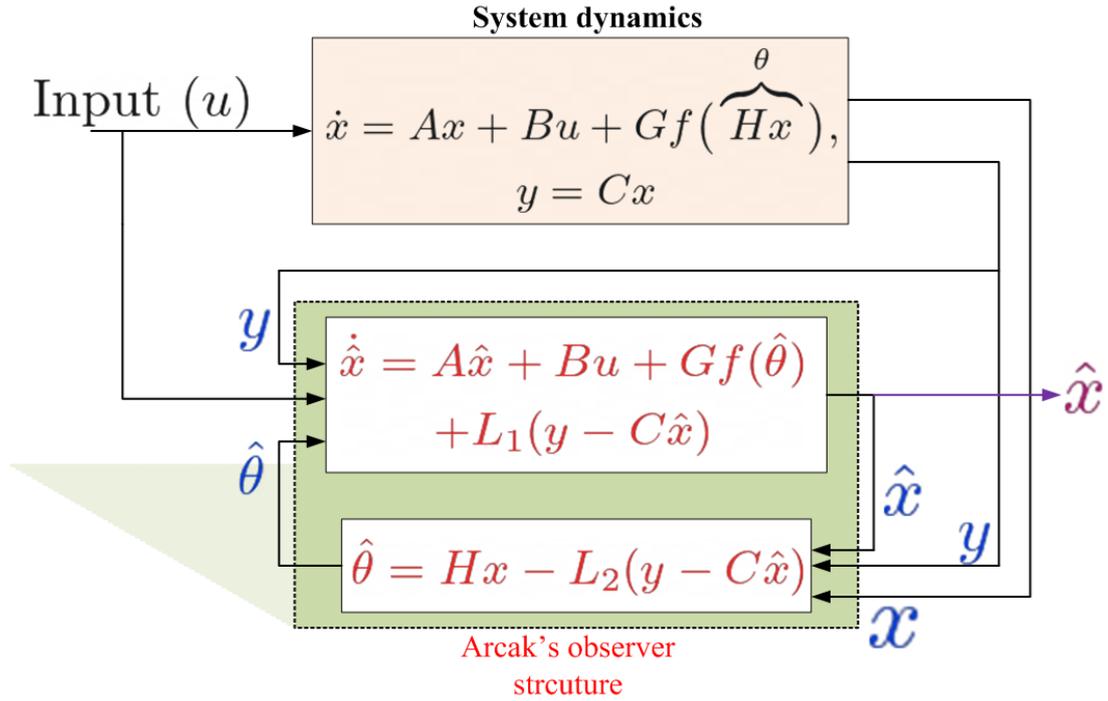


Figure 1.6: Diagrammatic representation of Arcak's observer

Further, authors of [75] relaxed the constraint (1.74) on  $f(Hx)$  by considering that it holds

$$0 \leq \frac{f_i(x_1) - f_i(x_2)}{x_1 - x_2} \leq b_i, \forall x_1 \neq x_2 \in \mathbb{R}, \quad (1.78)$$

where each  $b_i$  is a positive scalar.

From (1.78), the ensuing LMI condition is deduced.

**Theorem 1.11:** [75]

If there exist a symmetric positive definite matrix  $P$ , a positive scalar  $\epsilon$  and a diagonal matrix  $\Lambda$  such that

$$\begin{bmatrix} (A - L_1C)^\top P + P(A - L_1C) + \epsilon \mathbb{I} & PG + (H - L_2C)^\top \Lambda \\ \star & -\frac{2}{b} \end{bmatrix} < 0 \quad (1.79)$$

Then, the estimation error dynamic (1.76) is exponentially stable at the origin.

Furthermore, Arcak's observers gained a lot of attention from researchers, and several enhanced results are presented in [13, 14, 22, 76, 77].

## 1.4 Conclusion

This chapter primarily provided a brief summary of the fundamental notions utilised in the development of nonlinear observers. The presentation offers a distinctive point of view on

the subject and makes no claim to be exhaustive. In particular, some important concepts of observability and stability are illustrated in the chapter. In addition to this, some mathematical tools required for the observer design were recalled. Further, a few observer design methodologies were discussed. It shows that there is no particular method for nonlinear observer design.

---

# Evolving the LMI-based nonlinear observers: matrix multipliers as a key element

*“Enhancement comes from exploring fresh methods, not merely perfecting existing ones.”*

*Unknown*

## Contents

---

<b>2.1</b>	<b>Introduction</b>	<b>36</b>
<b>2.2</b>	<b>Towards enhanced LMI-based observers: hurdles and potentialities</b>	<b>36</b>
<b>2.3</b>	<b>Preliminaries</b>	<b>39</b>
<b>2.4</b>	<b>Incorporating the matrix multipliers in LMI-based nonlinear observers</b>	<b>42</b>
2.4.1	Problem formulation	42
2.4.2	Synthesis of LMI condition	45
2.4.3	Comment about the proposed LMI condition	47
<b>2.5</b>	<b>Enhancement in the matrix multiplier-based LMI approach</b>	<b>47</b>
2.5.1	Problem Contextualization	47
2.5.2	LMI design with Young inequality	50
2.5.3	LPV-based LMI approach	52
2.5.4	LMIs for the nonlinear systems with linear outputs	53
<b>2.6</b>	<b>Validation of proposed methodologies</b>	<b>54</b>
2.6.1	Numerical example 1	54
2.6.2	Numerical example 2	55
2.6.3	Implementation on a robotic arm manipulator	56
<b>2.7</b>	<b>Conclusion</b>	<b>58</b>

---

## 2.1 Introduction

Observer design for nonlinear systems has garnered a lot of attention in recent years. This is because comprehending the system's current state is crucial in numerous applications. Various tasks in the control domain, such as monitoring the systems, controlling the systems, and decision-making, are performed through the utilisation of such real-time information. Consequently, observers serve an integral role in modern-day applications, for example, autonomous vehicle tracking [15], the state-of-charge estimation of LI-ion battery model [68], cardiovascular application [78], and so on. The effectiveness of observers for linear systems has been thoroughly investigated. The first observer approach developed for linear systems was the Luenberger observer [39]. In comparison to linear systems, observer design for nonlinear systems is still an arduous problem. There seems to be no standardised design process for the development of observers in nonlinear systems. Due to this, an abundant amount of research has been conducted in this discipline, and numerous methods have been developed for each type of system (for example, [75], [76], [79], [56], [80], etc.). In the most recent literature, several novel observer design techniques have been proffered in [81] [22], [82], [83] and [84]. Among these techniques, LMI-based methodologies have earned an ample amount of interest and several outcomes have been published in [12], [13], [21], [14], [22] and [15]. Some of these approaches depended on S-Procedure lemma [18], Riccati equations [11], and Young inequality [17]. Though each method provides a unique way to tackle nonlinearities, there is scope for improvement.

The objective of the chapter is to introduce a novel LMI condition that can be enhanced further to provide a less conservative LMI method than existing ones. The remainder of the chapter is organised as follows: An overview of LMI-based observers and the motivation of the proposed methodology are discussed in Section 2.2. The prerequisites required for new observer design methods are illustrated in Section 2.3. A new matrix multiplier-based LMI approach is presented in 2.4. Further, Section 2.5 contains an enhanced matrix multiplier-based LMI approach. Section 2.6 includes the examination of the proposed techniques through numerical examples. At the end, some conclusions are given in Section 2.7.

## 2.2 Towards enhanced LMI-based observers: hurdles and potentialities

As stated in the aforementioned section, there has been an enormous amount of research conducted on the topic of LMI-based nonlinear observers. Amidst these methods, a few design techniques are summarized in Table 2.1. Several of these methods are recapitulated in Section 1.3. In this section, additional popular approaches are revisited to showcase the possible enhancements of the existing LMI-based techniques.

Table 2.1: Summary of LMI-based nonlinear observer approaches

No	Method	Citation
1	Observers based on Lipschitz properties	[14], [21], [11]
2	Observers based on bounded Jacobian	[83], [85]
3	Observers for systems with nonlinearities satisfying $\delta QC$	[77], [86]
4	Observers for systems with non-monotonic nonlinearities	[15]

This section begins with a discussion of the features of the method proposed in [21]. In this

chapter, the authors presented the LMI-based nonlinear observer by utilising a reformulated Lipschitz property (see Definition 3 of Appendix A.1 and the well-known LPV approach.

In [21], the following class of a nonlinear system is used:

$$\begin{aligned}\dot{x} &= Ax + G\gamma(x) + Bu, \\ y &= Cx,\end{aligned}\tag{2.1}$$

where  $x \in \mathbb{R}^n$  is the state vector,  $y \in \mathbb{R}^p$  is the output measurement and the input of the system is denoted by  $u \in \mathbb{R}^s$ . The matrices  $A \in \mathbb{R}^{n \times n}$ ,  $G \in \mathbb{R}^{n \times m}$ ,  $B \in \mathbb{R}^{n \times s}$ , and  $C \in \mathbb{R}^{p \times n}$  are known and constant. The function  $\gamma(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is assumed to be globally Lipschitz. The subsequent observer structure is employed for the state estimation purpose:

$$\dot{\hat{x}} = A\hat{x} + G\gamma(\hat{x}) + Bu + L(y - C\hat{x}),\tag{2.2}$$

where  $\hat{x}$  is the estimated state obtained from the observer, and  $L$  is a gain matrix. The estimation error of the observer is defined as  $\tilde{x} = x - \hat{x}$ . From (2.1) and (2.2), the error dynamic is given by

$$\dot{\tilde{x}} = (A - LC)\tilde{x} + G(\gamma(x) - \gamma(\hat{x})).\tag{2.3}$$

Further, by implementing the reformulated Lipschitz property (A.6) on  $f(x)$ , we get

$$\gamma(x) - \gamma(\hat{x}) = \sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} \mathcal{H}_{ij},\tag{2.4}$$

$$\gamma_{ij_{\min}} \leq \gamma_{ij} \leq \gamma_{ij_{\max}}.\tag{2.5}$$

For more details about this, one can refer to [21]. Therefore, the error dynamic (2.3) is modified as

$$\dot{\tilde{x}} = (A - LC)\tilde{x} + G\left(\sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} \mathcal{H}_{ij}\right).\tag{2.6}$$

The stability condition of the error dynamic (2.6) is obtained by utilising the LPV approach and Lyapunov theory. The derived criterion is and stated as follows:

**Theorem 2.1:** [21]

The error dynamic (2.3) is asymptotically stable if there exist matrices  $P > 0 \in \mathcal{S}^n$  and  $R \in \mathbb{R}^{p \times n}$  so that the ensuing optimization problem is solvable:

$$A(\Phi)^\top P + PA(\Phi) + A^\top P + PA - C^\top R - R^\top C \leq 0, \forall \Phi \in \mathcal{V}_{H_n},\tag{2.7}$$

where

$$A(\Phi) = \sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} G \mathcal{H}_{ij},\tag{2.8}$$

$$\mathcal{V}_{H_n} = \{\Phi \in \mathbb{R}^{n \times n} : \phi_{ij} \in \{\gamma_{ij_{\min}}, \gamma_{ij_{\max}}\}\}.\tag{2.9}$$

The observer gain is computed by using  $L = P^{-1}R^\top$ .

**Proof:** For the proof, one can follow the steps provided in [21]. □

The positive aspects of this method are enumerated below:

- I. LMI (2.7) provides a feasible solution for the larger Lipschitz constants, whereas, for the same constants, the LMI conditions of [11] and [75] are infeasible. It interprets that the LMI of [21] is more efficient than the one proposed in [11] and [75].
- II. Another advantage of the aforementioned approach is that the use of the newly reformulated Lipschitz condition allows the introduction of additional degrees of freedom than those obtained from the standard Lipschitz property.

Further, the authors of [14] and [17] had incorporated a new variant of Young inequality (A.10) with the reformulated Lipschitz property and derived a new LMI condition for  $\mathcal{H}_\infty$  nonlinear observers. In [17], the authors considered the subsequent nonlinear system:

$$\begin{aligned} \dot{x} &= Ax + G\gamma(x) + Bu + E\omega, \\ y &= Cx + D\omega, \end{aligned} \quad (2.10)$$

where  $\omega \in \mathbb{R}^q$  is  $\mathcal{L}_2$  bounded disturbance affecting both system dynamics and measurements.  $E \in \mathbb{R}^{n \times q}$  and  $D \in \mathbb{R}^{p \times q}$  are known constant matrices. Other parameters are the same as those defined in (2.1). The function  $\gamma(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is presumed to be a global Lipschitz nonlinear function, and it is represented in the following form:

$$\gamma(x) = \begin{bmatrix} \gamma_1(H_1x) \\ \gamma_2(H_2x) \\ \vdots \\ \underbrace{\gamma_i(H_ix)}_{\nu_i} \\ \vdots \\ \gamma_m(H_mx) \end{bmatrix}, \quad (2.11)$$

where  $H_i \in \mathbb{R}^{n_i \times n}$ . For the state estimation of the nonlinear system (2.10), an observer structure specified in (2.2) is used. From (2.10) and (2.2), the error dynamic is computed as

$$\dot{\tilde{x}} = (A - LC)\tilde{x} + G(\gamma(x) - \gamma(\hat{x})) + (E - LD)\omega. \quad (2.12)$$

Furthermore, analogous to (2.4), we get:

$$\gamma(x) - \gamma(\hat{x}) = \sum_{i=1}^m \sum_{j=1}^{n_i} \gamma_{ij} \mathcal{H}_{ij} H_i (x - \hat{x}), \quad (2.13)$$

where  $\mathcal{H}_{ij} = e_n(i)^\top e_n(j)$ . In addition to this, each function  $\gamma_{ij}$  fulfils (2.5). Hence, the error dynamic (2.12) is reformulated as

$$\dot{\tilde{x}} = (A - LC)\tilde{x} + \sum_{i=1}^m \sum_{j=1}^{n_i} (\gamma_{ij} G \mathcal{H}_{ij} H_i \tilde{x}) + (E - LD)\omega. \quad (2.14)$$

The following theorem provides an LMI condition for the asymptotic stability of the error dynamics (2.14):

**Theorem 2.2:** [17]

The estimation error dynamic (2.14) is  $\mathcal{H}_\infty$  asymptotically stable if there exist  $P > 0 \in \mathcal{S}^n$ ,  $Z_{ij} > 0 \in \mathcal{S}^{n_i} \forall i \in \{1, \dots, m\}, j \in \{1, \dots, n_i\}$  and  $R \in \mathbb{R}^{p \times n}$  such that the ensuing convex optimization problem is solvable:

$$\text{minimize } \mu, \quad (2.15)$$

$$\text{subject to } \begin{bmatrix} \mathcal{A} & \Sigma \\ \star & -\Lambda \mathbb{Z} \end{bmatrix} < 0, \quad (2.16)$$

where

$$\mathcal{A} = \begin{bmatrix} A^\top P + PA - R^\top C - C^\top R + \mathbb{I}_n & PE^\top - R^\top D \\ \star & -\mu \mathbb{I}_q \end{bmatrix}, \quad (2.17)$$

$$\Sigma = [\Sigma_1 \quad \Sigma_2 \quad \dots \quad \Sigma_m], \quad (2.18)$$

$$\Sigma_i = [\mathcal{N}_i^1(P, Z_{i1}) \quad \mathcal{N}_i^2(P, Z_{i2}) \quad \dots \quad \mathcal{N}_i^{n_i}(P, Z_{in_i})], \quad (2.19)$$

$$\mathcal{N}_i^j(P, Z_{ij}) = \begin{bmatrix} PG\mathcal{H}_{ij} \\ \mathbb{O} \end{bmatrix} + \begin{bmatrix} H_i^\top \\ \mathbb{O} \end{bmatrix} Z_{ij}, \quad (2.20)$$

$$\Lambda = \text{block-diag} \left( \frac{2}{\gamma_{11\max}} \quad \dots \quad \frac{2}{\gamma_{1n_1\max}} \quad \dots \quad \frac{2}{\gamma_{m1\max}} \quad \dots \quad \frac{2}{\gamma_{mn_m\max}} \right), \quad (2.21)$$

$$\mathbb{Z} = \text{block-diag} (Z_{11} \quad \dots \quad Z_{1n_1} \quad \dots \quad Z_{m1} \quad \dots \quad Z_{mn_m}). \quad (2.22)$$

**Proof:** For the proof, one can follow the development presented in [17] by considering  $\alpha = 0$ .  $\square$

In [17] and [14], the authors had developed a less conservative LMI condition than the one in [12] and [13] for the nonlinear Lipschitz systems. The key point of this method is **the inclusion of a block-diagonal matrix** along with **a new variant of Young inequality** in the LMI formulation. It yields an enhanced LMI condition with additional numbers of degrees of freedom from a feasibility point of view. The addition of a block-diagonal multiplier matrix in the LMI approach is intriguing and beneficial, yet there is still scope for enhancement. **The question that arises here is why a block-diagonal matrix structure only.**

This question inspires the authors to propose a novel LMI condition based on a more generalised matrix multiplier than a block diagonal matrix. The inclusion of such a matrix may include additional decision variables inside the LMI condition and may lead to an enhancement in existing LMI approaches. The motivation behind the development of the new LMI condition is depicted in Figure 2.1.

Before presenting the novel LMI conditions, some preliminaries are illustrated in the next section.

## 2.3 Preliminaries

This section is devoted to the introduction of some prerequisites that will be utilised in this chapter and the subsequent chapters.

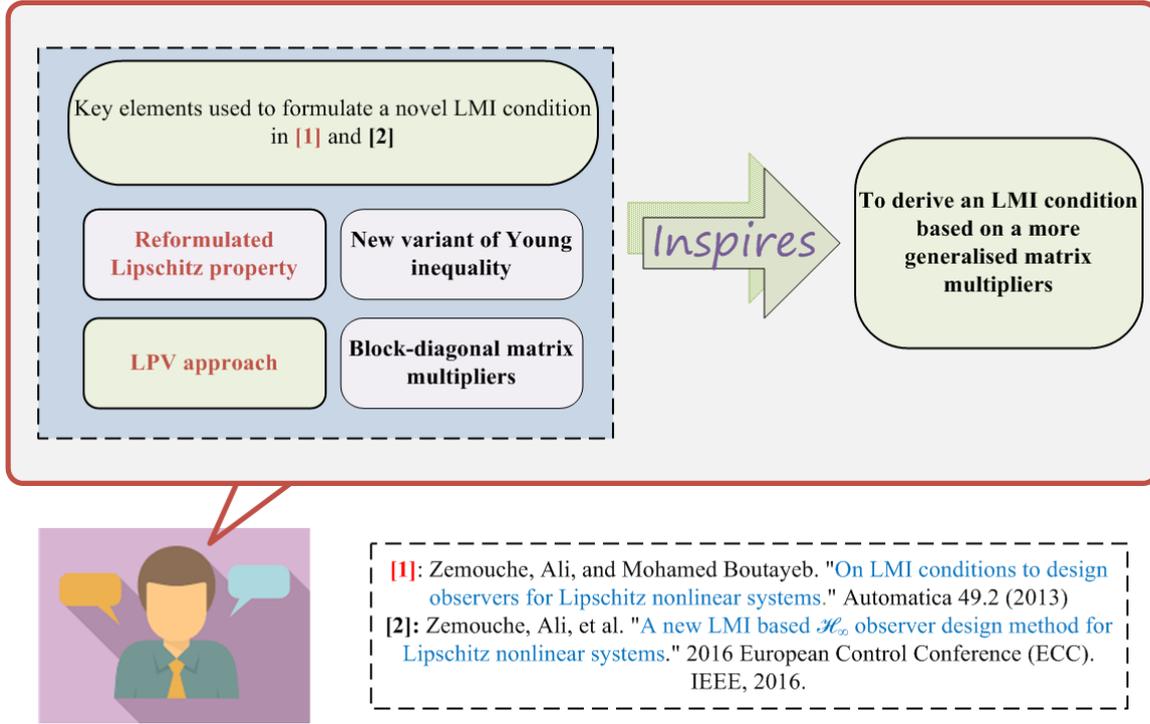


Figure 2.1: Encouraging towards the development of new matrix multiplier-based LMIs

**Lemma 2.1:** [21]

Let  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a nonlinear function. Then, the following two statements are equivalent:

1.  $\phi$  is globally Lipschitz with respect to its argument, i.e.,

$$\|\phi(X) - \phi(Y)\| \leq \phi_\gamma \|X - Y\|, \quad \forall X, Y \in \mathbb{R}^n. \quad (2.23)$$

2. For all,  $i, j = 1, \dots, n$ , there exist functions  $\phi_{ij} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ , and constants  $\phi_{ij_{\min}}$  and  $\phi_{ij_{\max}}$  such that  $\forall X, Y \in \mathbb{R}^n$ ,

$$\phi(X) - \phi(Y) = \sum_{i=1}^n \sum_{j=1}^n \phi_{ij} \mathcal{H}_{ij}(X - Y), \quad (2.24)$$

where  $\mathcal{H}_{ij} = e_n(i)e_n^\top(j)$ , and  $\phi_{ij} \triangleq \phi_{ij}(X_i^{Y_{i,j-1}}, X_i^{Y_{i,j}})$ .

The functions  $\phi_{ij}(\cdot)$  are globally bounded as follows:

$$\phi_{ij_{\min}} \leq \phi_{ij} \leq \phi_{ij_{\max}}. \quad (2.25)$$

One can go through Appendix A.1 for more details about Lemma 2.1 .

**Lemma 2.2**

Consider the ensuing matrices:

$$\Psi_a^\top = \begin{bmatrix} a_1 \mathbb{I}_n & a_2 \mathbb{I}_n & \dots & a_n \mathbb{I}_n \end{bmatrix}, \quad (2.26)$$

$$\Psi_b^\top = \begin{bmatrix} b_1 \mathbb{I}_n & b_2 \mathbb{I}_n & \dots & b_n \mathbb{I}_n \end{bmatrix}, \quad (2.27)$$

where  $0 \leq a_i \leq b_i \forall i \in \{1, \dots, n\}$ .

Let us define

$$Z = \begin{bmatrix} Z_1 & Z_{a_1^2} & \dots & Z_{a_1^{n-1}} & Z_{a_1^n} \\ \star & Z_2 & \dots & Z_{a_2^{n-1}} & Z_{a_2^n} \\ \star & \star & \ddots & \vdots & \vdots \\ \star & \star & \dots & Z_{n-1} & Z_{a_{n-1}^n} \\ \star & \star & \dots & \star & Z_n \end{bmatrix}, \quad (2.28)$$

where  $Z_i = Z_i^\top > 0 \in \mathbb{R}^{n \times n}$ ,  $Z_{a_i^j} = Z_{a_i^j}^\top \geq 0 \in \mathbb{R}^{n \times n} \forall i \in \{1, \dots, n\}$  so that  $Z > 0$ .

Then, the succeeding inequality is true:

$$\Psi_a^\top Z \Psi_a \leq \Psi_b^\top Z \Psi_b. \quad (2.29)$$

**Proof:** By using (2.26), (2.27) and (2.28),  $\Psi_a^\top Z \Psi_a$  and  $\Psi_b^\top Z \Psi_b$  are computed as follows:

$$\begin{aligned} \Psi_a^\top Z \Psi_a &= a_1^2 Z_1 + a_2^2 Z_2 + \dots + a_n^2 Z_n + 2a_1 a_2 Z_{a_1^2} + \dots + 2a_1 a_n Z_{a_1^n} + 2a_2 a_3 Z_{a_2^3} + \dots \\ &\quad + 2a_2 a_n Z_{a_2^n} + 2a_3 a_4 Z_{a_3^4} + \dots + 2a_{n-1} a_n Z_{a_{n-1}^n}, \end{aligned}$$

$$\begin{aligned} \Psi_b^\top Z \Psi_b &= b_1^2 Z_1 + b_2^2 Z_2 + \dots + b_n^2 Z_n + 2b_1 b_2 Z_{a_1^2} + \dots + 2b_1 b_n Z_{a_1^n} + 2b_2 b_3 Z_{a_2^3} + \dots \\ &\quad + 2b_2 b_n Z_{a_2^n} + 2b_3 b_4 Z_{a_3^4} + \dots + 2b_{n-1} b_n Z_{a_{n-1}^n}. \end{aligned}$$

Then,

$$\begin{aligned} \Psi_a^\top Z \Psi_a - \Psi_b^\top Z \Psi_b &= \underbrace{(a_1^2 - b_1^2)}_{\Delta_1} Z_1 + \underbrace{(a_2^2 - b_2^2)}_{\Delta_2} Z_2 + \dots + \underbrace{(a_n^2 - b_n^2)}_{\Delta_n} Z_n + 2 \underbrace{(a_1 a_2 - b_1 b_2)}_{\Delta_{11}} Z_{a_1^2} \\ &\quad + \dots + 2 \underbrace{(a_1 a_n - b_1 b_n)}_{\Delta_{1(n-1)}} Z_{a_1^n} + 2 \underbrace{(a_2 a_3 - b_2 b_3)}_{\Delta_{21}} Z_{a_2^3} + \dots \\ &\quad + 2 \underbrace{(a_2 a_n - b_2 b_n)}_{\Delta_{2(n-2)}} Z_{a_2^n} + \dots + 2 \underbrace{(a_{n-1} a_n - b_{n-1} b_n)}_{\Delta_{(n-1)1}} Z_{a_{n-1}^n}. \end{aligned}$$

Since  $0 \leq a_i \leq b_i \forall i \in \{1, \dots, n\}$ ,

$$\begin{aligned} \Delta_i &\leq 0 \forall i \in \{1, \dots, n\} \\ \Delta_{ij} &\leq 0 \forall i, j \in \{1, \dots, n-1\}. \end{aligned}$$

Further, from  $Z_i > 0$  and  $Z_{a_i} \geq 0$ , we get:

$$\Psi_a^\top Z \Psi_a - \Psi_b^\top Z \Psi_b \leq 0.$$

Hence, proved.  $\square$

**Corollary 2.1**

If there exist matrices  $\Psi_\phi$  and  $\Psi_b$ , illustrated in (2.26) and (2.26), respectively, along with the matrix  $Z$  in the following form:

$$Z = \begin{bmatrix} Z_1 & \alpha Z_2 & \dots & \alpha Z_n \\ \star & Z_2 & \dots & \alpha Z_n \\ \star & \star & \ddots & \alpha Z_n \\ \star & \star & \star & Z_n \end{bmatrix}, \quad (2.30)$$

where  $Z_i = Z_i^\top > 0 \forall i \in \{1, \dots, n\}$ , and  $0 \leq \alpha \leq 1$ . Then, the following inequality fulfils:

$$\Psi_a^\top Z \Psi_a \leq \Psi_b^\top Z \Psi_b. \quad (2.31)$$

**Proof:** In the proof of Lemma 2.2, if  $Z_{a_i^j}$  is replaced by  $\alpha Z_i$ , it is easy to deduce the inequality (2.31).  $\square$

**Corollary 2.2**

If there exist matrices  $\Psi_a$ ,  $\Psi_b$  and  $Z$  defined in (2.26), (2.27) and (2.28) respectively, and both fulfils the inequality (2.29). Then, for any block-diagonal matrix  $A$  of appropriate dimensions, the subsequent inequality holds true:

$$\Psi_a^\top A^\top Z A \Psi_a \leq \Psi_b^\top A^\top Z A \Psi_b. \quad (2.32)$$

**Proof:** For any block-diagonal matrix  $A$ , if matrix  $Z$  is positive definite, then matrix  $A^\top Z A$  is also positive semi-definite. Thus, by applying Lemma 2.2 on  $\Psi_a$  and  $\Psi_b$  with a positive semi-definite matrix  $A^\top Z A$ , the inequality (2.32) is achieved, which ends proof.  $\square$

## 2.4 Incorporating the matrix multipliers in LMI-based nonlinear observers

This section is focused on an attempt to address the question raised at the end of Section 2.2. In the corresponding section, an LMI condition is derived by using Young inequality, the reformulated Lipschitz property, and a newly defined matrix structure.

### 2.4.1 Problem formulation

For simplicity of the presentation, the system with nonlinear dynamics and linear outputs is considered for observer development and it is portrayed by the following set of equations:

$$\begin{aligned} \dot{x} &= Ax + Gf(x) + Bu, \\ y &= Cx, \end{aligned} \quad (2.33)$$

where  $x \in \mathbb{R}^n$ ,  $y \in \mathbb{R}^p$  and  $u \in \mathbb{R}^s$  represent the states, the output measurements, and the input of the system, respectively.  $A \in \mathbb{R}^{n \times n}$ ,  $G \in \mathbb{R}^{n \times m}$ ,  $B \in \mathbb{R}^{n \times s}$ , and  $C \in \mathbb{R}^{p \times n}$  are known constant

matrices. The function  $f(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is assumed to be globally Lipschitz, and it is written under the succeeding form:

$$f(x) = \begin{bmatrix} f_1(H_1x) \\ f_2(H_2x) \\ \vdots \\ \underbrace{f_i(H_ix)}_{\nu_i} \\ \vdots \\ f_m(H_mx) \end{bmatrix}, \quad (2.34)$$

where  $H_i \in \mathbb{R}^{n_i \times n}$ . Without loss of generality, let us presume that  $n_i$  is the same for all the nonlinear functions  $f_1, \dots, f_m$ , i.e.,  $n_i = \bar{n}$ ,  $\forall i = 1, \dots, m$ . Since matrices  $H_i$  are not unique, this gives the freedom of choosing  $\bar{n}$  accordingly which is the benefit of the proposed method over the method of [76]. The choice of  $\bar{n}$  plays a vital role in the improvements of the new LMI condition, which is highlighted in the sequel.

The ensuing Luenberger state observer is utilised for state estimation purposes:

$$\dot{\hat{x}} = A\hat{x} + G \begin{bmatrix} f_1(H_1\hat{x}) \\ f_2(H_2\hat{x}) \\ \vdots \\ \underbrace{f_i(H_i\hat{x})}_{\hat{\nu}_i} \\ \vdots \\ f_m(H_m\hat{x}) \end{bmatrix} + Bu + L(y - C\hat{x}), \quad (2.35)$$

where  $\hat{x}$  is the estimate of state  $x$  and the matrix  $L \in \mathbb{R}^{n \times p}$  is an observer gain. The estimation error of the observer (2.35) is defined as

$$\tilde{x} = x - \hat{x}. \quad (2.36)$$

Further, from (2.33) and (2.35), the estimation error dynamic is given by

$$\dot{\tilde{x}} = \mathbb{A}\tilde{x} + G(\tilde{f}(x, \hat{x})), \quad (2.37)$$

where

$$\mathbb{A} = A - LC, \quad (2.38)$$

$$\tilde{f}(x, \hat{x}) = f(x) - f(\hat{x}) = \begin{bmatrix} f_1(\nu_1) \\ f_2(\nu_2) \\ \vdots \\ f_m(\nu_m) \end{bmatrix} - \begin{bmatrix} f_1(\hat{\nu}_1) \\ f_2(\hat{\nu}_2) \\ \vdots \\ f_m(\hat{\nu}_m) \end{bmatrix}. \quad (2.39)$$

Since  $f(\cdot)$  is globally Lipschitz function, then from Lemma 2.1, there exist functions

$$f_{ij} : \mathbb{R}^{\bar{n}} \times \mathbb{R}^{\bar{n}} \rightarrow \mathbb{R} \text{ for } i \in \{1, \dots, m\} \text{ and } j \in \{1, \dots, \bar{n}\},$$

and constants  $f_{a_{ij}}$  and  $f_{b_{ij}}$ , such that,

$$\tilde{f}(x, \hat{x}) = \sum_{i,j=1}^{m,\bar{n}} f_{i,j} \mathcal{H}_{i,j} H_i \tilde{x}, \quad (2.40)$$

$$f_{a_{ij}} \leq f_{ij} \leq f_{b_{ij}}, \quad (2.41)$$

where  $f_{ij} \triangleq f_{ij}(\nu_i^{\hat{\nu}_{i,j-1}}, \nu_i^{\hat{\nu}_{i,j}})$  and  $\mathcal{H}_{i,j} = e_m(i)e_n^\top(j)$ .

Without loss of generality, let us assume that  $f_{a_{ij}} = 0 \forall i \in \{1, \dots, m\}$  and  $j \in \{1, \dots, \bar{n}\}$ , and it leads to:

$$0 \leq f_{ij} \leq f_{b_{ij}}. \quad (2.42)$$

For more details about this, one can refer to [12] and [21].

Through the use of (2.40), the error dynamic (2.37) is reformulated as

$$\dot{\tilde{x}} = \mathbb{A}\tilde{x} + \sum_{i,j=1}^{m,\bar{n}} f_{ij} G \mathcal{H}_{ij} H_i \tilde{x}. \quad (2.43)$$

### Objective

The primary goal of this section is to determine the gain matrix  $L$  such that the error dynamic (2.43) is exponentially stable.

To obtain the necessary condition, the following standard quadratic Lyapunov function is deployed:

$$V(\tilde{x}) = \tilde{x}^\top P \tilde{x}, \text{ where } P > 0 \in \mathbf{S}^n.$$

Thus,  $V(\tilde{x})$  holds:

$$\lambda_{\min}(P) \|\tilde{x}\|^2 \leq V(\tilde{x}) \leq \lambda_{\max}(P) \|\tilde{x}\|^2. \quad (2.44)$$

The essential criterion for the exponential stability of a system is illustrated in Theorem 1.2. The condition (2.44) is identical to the one specified in (1.26). To achieve an exponential convergence of  $\tilde{x}$  towards zero, another necessary condition on  $\dot{V}(\tilde{x})$  is described as follows:

$$\dot{V}(\tilde{x}) + \epsilon \|\tilde{x}\|^2 \leq 0, \text{ where } \epsilon \text{ is a positive scalar.} \quad (2.45)$$

The inequality (2.45) is equivalent to the expression (1.27). The derivative of  $V(\tilde{x})$  along the trajectories of (2.43) is given by

$$\dot{V}(\tilde{x}) = \tilde{x}^\top P \left( \mathbb{A}\tilde{x} + \sum_{i,j=1}^{m,\bar{n}} f_{ij} G \mathcal{H}_{ij} H_i \tilde{x} \right) + \left( \mathbb{A}\tilde{x} + \sum_{i,j=1}^{m,\bar{n}} f_{ij} G \mathcal{H}_{ij} H_i \tilde{x} \right)^\top P \tilde{x}.$$

Hence,

$$\dot{V}(\tilde{x}) = \tilde{x}^\top \left( \mathbb{A}^\top P + P \mathbb{A} \right) \tilde{x} + \tilde{x}^\top \left( \left( \sum_{i,j=1}^{m,\bar{n}} f_{ij} P G \mathcal{H}_{ij} H_i \right) + \left( \sum_{i,j=1}^{m,\bar{n}} f_{ij} P G \mathcal{H}_{ij} H_i \right)^\top \right) \tilde{x}. \quad (2.46)$$

From (2.45) and (2.46), the required condition for obtaining the exponential stability of the error dynamic (2.43) is

$$\left( \mathbb{A}^\top P + P \mathbb{A} + \epsilon \mathbb{I} \right) + \underbrace{\left( \left( \sum_{i,j=1}^{m,\bar{n}} f_{ij} P G \mathcal{H}_{ij} H_i \right) + \left( \sum_{i,j=1}^{m,\bar{n}} f_{ij} P G \mathcal{H}_{ij} H_i \right)^\top \right)}_{N_1} \leq 0. \quad (2.47)$$

In the literature, several LMI-based methods have been developed for this class of systems. All the proposed approaches provided LMI relaxations; however, the resulting LMI conditions

remain conservative, and further improvements are possible. This is the motivation for proposing new LMI conditions.

In the next subsection, Lemma 2.2 and Young inequality (A.9) will be exploited to improve the standard LMI methods. In this approach, a matrix multiplier will be incorporated in the LMI conditions, providing more degrees of freedom from a feasibility perspective compared to existing methods.

### 2.4.2 Synthesis of LMI condition

Over the last two decades, various LMI-based techniques were proposed to address the Lipschitz nonlinearity. Some classical methods rely on the S-Procedure lemma [11] or the standard Young inequality [32]. In all these approaches, the authors tackled the nonlinearities with the standard Lipschitz property (A.1), rather than using reformulated Lipschitz property (A.6). In this subsection, a new LMI condition is designed by using Lemma 2.2 and an inequality (A.9). The use of the detailed form allows the inclusion of more decision variables than methods using the global form. Such extra degrees of freedom enhance the LMI feasibility.

In the proposed technique, the nonlinearity is handled differently as compared to [14]. The nonlinear term  $N_1$  of (2.47) is written under the following form:

$$N_1 = \underbrace{\begin{bmatrix} f_{11}\mathbb{I}_n & \cdots & f_{1\bar{n}}\mathbb{I}_n & \cdots & f_{m1}\mathbb{I}_n & \cdots & f_{m\bar{n}}\mathbb{I}_n \end{bmatrix}}_{\Phi^\top} \underbrace{\begin{bmatrix} (PG\mathcal{H}_{11}H_1)^\top \\ \vdots \\ (PG\mathcal{H}_{1\bar{n}}H_1)^\top \\ \vdots \\ (PG\mathcal{H}_{m1}H_m)^\top \\ \vdots \\ (PG\mathcal{H}_{m\bar{n}}H_m)^\top \end{bmatrix}}_{\bar{U}} + \bar{U}^\top \Phi. \quad (2.48)$$

By implementing Young inequality (A.9) on (2.48), we get:

$$N_1 = \Phi^\top \bar{U} + \bar{U}^\top \Phi \leq \bar{U}^\top \mathbb{Z}^{-1} \bar{U} + \Phi^\top \mathbb{Z} \Phi, \quad (2.49)$$

where  $\mathbb{Z} > 0$ , and it is illustrated as follows:

$$\mathbb{Z} = \begin{bmatrix} Z_{11} & \alpha Z_{12} & \cdots & \alpha Z_{1\bar{n}} & \cdots & \alpha Z_{m1} & \cdots & \alpha Z_{m\bar{n}} \\ \alpha Z_{12} & Z_{12} & \cdots & \alpha Z_{1\bar{n}} & \cdots & \alpha Z_{m1} & \cdots & \alpha Z_{m\bar{n}} \\ \star & \star & \ddots & \alpha Z_{1\bar{n}} & \cdots & \alpha Z_{m1} & \cdots & \alpha Z_{m\bar{n}} \\ \star & \star & \star & Z_{1\bar{n}} & \cdots & \alpha Z_{m1} & \cdots & \alpha Z_{m\bar{n}} \\ \star & \star & \star & \star & \ddots & \alpha Z_{m1} & \cdots & \alpha Z_{m\bar{n}} \\ \star & \star & \star & \star & \star & Z_{m1} & \cdots & \alpha Z_{m\bar{n}} \\ \star & \star & \star & \star & \star & \star & \ddots & \alpha Z_{m\bar{n}} \\ \star & Z_{m\bar{n}} \end{bmatrix}, \quad (2.50)$$

where  $Z_{ij} > 0 \in \mathbf{S}^n, \forall i \in \{1, \dots, m\}; j \in \{1, \dots, \bar{n}\}$ , and  $0 \leq \alpha \leq 1$ .

Let us define

$$\Phi_m = \begin{bmatrix} f_{b_{11}}\mathbb{I}_n & \cdots & f_{b_{1\bar{n}}}\mathbb{I}_n & \cdots & f_{b_{m1}}\mathbb{I}_n & \cdots & f_{b_{m\bar{n}}}\mathbb{I}_n \end{bmatrix}^\top \quad (2.51)$$

Since  $0 \leq f_{ij} \leq f_{bij}$ , the following inequality is derived by using Corollary 2.1 on the matrices  $\Phi$ ,  $\Phi_m$  and  $\mathbb{Z}$ :

$$\Phi^\top \mathbb{Z} \Phi - \Phi_m^\top \mathbb{Z} \Phi_m \leq 0. \quad (2.52)$$

Thus, the inequality (2.49) is modified as

$$N_1 \leq \bar{U}^\top \mathbb{Z}^{-1} \bar{U} + \Phi_m^\top \mathbb{Z} \Phi_m. \quad (2.53)$$

Hence, the necessary condition (2.47) is satisfied, if

$$\left( \mathbb{A}^\top P + P \mathbb{A} + \epsilon \mathbb{I} \right) + \bar{U}^\top \mathbb{Z}^{-1} \bar{U} + \Phi_m^\top \mathbb{Z} \Phi_m \leq 0 \quad (2.54)$$

Now, we can proceed to establish the subsequent theorem.

### Theorem 2.3

The estimation error dynamic (2.43) converges exponentially towards zero if there exist scalars  $\epsilon > 0$ , and  $0 \leq \alpha \leq 1$  along with matrices  $P > 0 \in \mathbf{S}^n$ ,  $\mathbb{Z}$  under the form of (2.50) and  $R \in \mathbb{R}^{p \times n}$ , such that the following LMI condition is feasible:

$$\begin{bmatrix} \mathbb{A}(A, P, R) & \bar{U}^\top & \Phi_m^\top \mathbb{Z} \\ \star & -\mathbb{Z} & \mathbb{O} \\ \star & \star & -\mathbb{Z} \end{bmatrix} < 0, \quad (2.55)$$

where

$$\mathbb{A}(A, P, R) = A^\top P + P A - R^\top C - C^\top R + \epsilon \mathbb{I}_n. \quad (2.56)$$

The matrices  $\bar{U}$  and  $\Phi_m$  are defined in (2.48) and (2.51), respectively. The observer gain is computed as  $L = P^{-1} R^\top$ .

**Proof:** Since we have (2.54), the proof is straightforward. Indeed, by using the change of variables  $R = L^\top P$ , and implementing the Schur lemma (A.3) on (2.54), LMI (2.55) is obtained. If the LMI condition (2.55) is feasible, then the exponential converges of error dynamics (2.43) towards zero is ensured.  $\square$

Table 2.2: Computation of the number of decision variables of LMI (2.55)

Number of decision variables inside element $\mathbb{A}(A, P, R)$	Number of decision variables inside matrix $\mathbb{Z}$	Total
$\underbrace{np}_R + \underbrace{\frac{n(n+1)}{2}}_P$	$m\bar{n} \binom{n(n+1)}{2} + \underbrace{1}_{\alpha^1}$ ( $^1$ : $\alpha$ is chosen by users)	$\frac{n(1+2p+n)}{2} + \underbrace{\left( m\bar{n} \binom{n(n+1)}{2} \right)}_{\text{additional variables}} + 1$

### 2.4.3 Comment about the proposed LMI condition

The objective of this subsection is to highlight the advantages of the proposed LMIs. In order to achieve more feasibility in convex optimization problems, one of the important parameters is the number of decision variables inside LMIs. Thus, the computation of this parameter for LMI (2.55) is showcased in Table 2.2. It interprets that the number of decision variables inside LMIs (2.55) is dependent on the system parameters  $n$  and  $\bar{n}$ . In the next section, the effectiveness of these parameters on the LMI feasibility is evaluated. The comparison of the number of additional decision variables with existing LMI methods is demonstrated in Table 2.3.

Table 2.3: Number of additional variables for several LMI methods

	LMI of [13]	LMI of [14]	LMI (2.55)
Number of additional variable	$\sum_{i=1}^m n_i$	$\sum_{i=1}^m \left( n_i \binom{n_i(n_i+1)}{2} \right)$	$m\bar{n} \binom{n(n+1)}{2} + 1$

It is easy to show that the number of additional variables inside the LMI (2.55) is higher than the existing LMIs of [13] and [14]. It is one of the advantages of the proposed LMI method over existing methods. This is due to the introduction of a new matrix multiplier in the LMI.

Though the proposed LMI includes an extra decision variable than existing approaches, these newly defined matrix variables are dependent upon variables  $n$  (i.e., number of states) which is restrictive in the case of large-scale dynamic systems. It may increase computational costs. In addition to this, the inclusion of this additional variable is unable to provide any substantial advancements in LMIs. Thus, there is a potential for improvements in the newly designed LMI method such as consideration of a more generalised matrix multiplier. In the sequel, the proposed LMI condition is enhanced by integrating a new matrix multiplier structure.

## 2.5 Enhancement in the matrix multiplier-based LMI approach

This section is dedicated to the development of an enhanced LMI condition based on a more generalised matrix multiplier than the one employed in Section 2.4. In this segment, a class of systems having nonlinear dynamics and nonlinear outputs is used for the observer design purpose. This case is often encountered in many industrial applications, for example, the problem of magnetic position estimation [87], the state of charge (SoC) estimation in battery management system [68], and automotive slip angle estimation problem [14].

### 2.5.1 Problem Contextualization

The following sets of equations represent a class of nonlinear systems with nonlinear outputs:

$$\begin{aligned} \dot{x} &= Ax + Gf(x) + Bu \\ y &= Cx + Fg(x), \end{aligned} \tag{2.57}$$

where  $F \in \mathbb{R}^{p \times r}$  is constant matrix. The function  $g(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^r$  is assumed to be globally Lipschitz, and it is written under the following form:

$$g(x) = \begin{bmatrix} g_1(G_1 x) \\ \vdots \\ \theta_i \\ g_i(\overbrace{G_i x}^{\theta_i}) \\ \vdots \\ g_r(G_r x) \end{bmatrix}, \quad (2.58)$$

where  $G_i \in \mathbb{R}^{\bar{p} \times n}$  is a constant matrix. The other matrices and variables remain the same as the ones specified in (2.33).

Consider the subsequent observer form:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Gf(\hat{x}) + Bu + L(y - \hat{y}), \\ \hat{y} &= C\hat{x} + Fg(\hat{x}). \end{aligned} \quad (2.59)$$

where  $\hat{x}$  is the estimated state, and  $L \in \mathbb{R}^{n \times p}$  is the observer gain matrix.

From (2.57) and (2.59), the error dynamic is computed as follows:

$$\dot{\tilde{x}} = \mathbb{A}\tilde{x} + G\tilde{f}(x, \hat{x}) - LF\tilde{g}(x, \hat{x}), \quad (2.60)$$

where  $\mathbb{A}$  and  $\tilde{f}(x, \hat{x})$  are described in (2.38) and (2.39), respectively.  $\tilde{g}(x, \hat{x})$  is given by

$$\tilde{g}(x, \hat{x}) = g(x) - g(\hat{x}) = \begin{bmatrix} g_1(G_1 x) \\ \vdots \\ \theta_i \\ g_i(\overbrace{G_i x}^{\theta_i}) \\ \vdots \\ g_r(G_r x) \end{bmatrix} - \begin{bmatrix} g_1(G_1 \hat{x}) \\ \vdots \\ \hat{\theta}_i \\ g_i(\overbrace{G_i \hat{x}}^{\hat{\theta}_i}) \\ \vdots \\ g_r(G_r \hat{x}) \end{bmatrix}. \quad (2.61)$$

Since  $g(\cdot)$  is a globally Lipschitz function, then from Lemma 2.1, there exist functions

$$g_{ij} : \mathbb{R}^{\bar{p}} \times \mathbb{R}^{\bar{p}} \rightarrow \mathbb{R}$$

and constants  $g_{a_{ij}}$  and  $g_{b_{ij}}$ , such that

$$\tilde{g}(x, \hat{x}) = \sum_{i,j=1}^{r,\bar{p}} g_{ij} \mathcal{G}_{ij}(G_i) \tilde{x}, \quad (2.62)$$

$$g_{a_{ij}} \leq g_{ij} \leq g_{b_{ij}}, \quad (2.63)$$

where  $g_{ij} \triangleq g_{ij}(\theta_i^{\hat{\theta}_{i,j-1}}, \theta_i^{\hat{\theta}_{i,j}})$  and  $\mathcal{G}_{ij} = e_r(i) e_{\bar{p}}^{\top}(j)$ . Without loss of generality, we assume that  $g_{a_{ij}} = 0$  for all  $i \in \{1, \dots, r\}$  and  $j \in \{1, \dots, \bar{p}\}$ , i.e.,

$$0 \leq g_{ij} \leq g_{b_{ij}}. \quad (2.64)$$

Hence, from (2.40) and (2.62), the error dynamic (2.60) is altered as:

$$\dot{\tilde{x}} = \mathbb{A}\tilde{x} + \sum_{i,j=1}^{m,\bar{n}} f_{ij} G \mathcal{H}_{ij} H_i \tilde{x} - \sum_{i,j=1}^{r,\bar{p}} g_{ij} L F \mathcal{G}_{ij} G_i \tilde{x}. \quad (2.65)$$

The aim of this section is to compute the gain matrix  $L$  which ensures the exponential stability of the error dynamic (2.65).

Let us consider the Lyapunov function  $V(\tilde{x}) = \tilde{x}^\top P \tilde{x}$ , where  $P > 0 \in \mathbf{S}^n$  for the stability analysis of the error dynamic (2.65). Analogous to Section 2.4,  $V(\tilde{x})$  fulfils the condition (2.44), which implies that  $V(\tilde{x})$  admits (1.26).

Further,  $\dot{V}(\tilde{x})$  is computed along the trajectories of (2.65), and illustrated as follows:

$$\begin{aligned} \dot{V}(\tilde{x}) &= \tilde{x}^\top \left( \mathbb{A}^\top P + P \mathbb{A} \right) \tilde{x} + \tilde{x}^\top \left( \left( \sum_{i,j=1}^{m,\bar{n}} f_{ij} P G \mathcal{H}_{ij} H_i \right) + \left( \sum_{i,j=1}^{m,\bar{n}} f_{ij} P G \mathcal{H}_{ij} H_i \right)^\top \right) \tilde{x} \\ &\quad - \tilde{x}^\top \left( \left( \sum_{i,j=1}^{r,\bar{p}} g_{ij} P L F \mathcal{G}_{ij} G_i \right) + \left( \sum_{i,j=1}^{r,\bar{p}} g_{ij} P L F \mathcal{G}_{ij} G_i \right)^\top \right) \tilde{x} \end{aligned}$$

Similar to Section 2.4, we deduce that the error dynamic (2.65) satisfies the exponential stability condition (2.45), if the following inequality holds:

$$\begin{aligned} &\left( \mathbb{A}^\top P + P \mathbb{A} + \epsilon \mathbb{I} \right) + \left( \left( \sum_{i,j=1}^{m,\bar{n}} f_{ij} P G \mathcal{H}_{ij} H_i \right) + \left( \sum_{i,j=1}^{m,\bar{n}} f_{ij} P G \mathcal{H}_{ij} H_i \right)^\top \right) \\ &\quad + \left( \left( \sum_{i,j=1}^{r,\bar{p}} g_{ij} (-P L F) \mathcal{G}_{ij} G_i \right) + \left( \sum_{i,j=1}^{r,\bar{p}} g_{ij} (-P L F) \mathcal{G}_{ij} G_i \right)^\top \right) \leq 0. \end{aligned} \quad (2.66)$$

By considering  $R^\top = P L$ , the inequality (2.66) is rewritten as:

$$\underbrace{\mathbb{A}(A, P, R) + \left[ \sum_{i,j=1}^{m,\bar{n}} \left( \underbrace{(P G \mathcal{H}_{ij})}_{\mathbb{U}_{ij}^\top} \overbrace{(f_{ij} H_i)}^{\mathbb{V}_{ij}} + \mathbb{V}_{ij}^\top \mathbb{U}_{ij} \right) \right]}_{N_1} + \underbrace{\left[ \sum_{i,j=1}^{r,\bar{p}} \left( \underbrace{(-R^\top F \mathcal{G}_{ij})}_{\mathbb{M}_{ij}^\top} \overbrace{(g_{ij} G_i)}^{\mathbb{N}_{ij}} + \mathbb{N}_{ij}^\top \mathbb{M}_{ij} \right) \right]}_{N_2} \leq 0, \quad (2.67)$$

where  $\mathbb{A}(A, P, R)$  is described in (2.56).

Now, the terms  $N_1$  and  $N_2$  are written under the following form:

$$N_1 = \mathbb{U}^\top \mathbb{V} + \mathbb{V}^\top \mathbb{U}, \quad (2.68)$$

$$N_2 = \mathbb{M}^\top \mathbb{N} + \mathbb{M}^\top \mathbb{N}, \quad (2.69)$$

where

$$\mathbb{U} = \left[ \mathbb{U}_{11}^\top \quad \cdots \quad \mathbb{U}_{1\bar{n}}^\top \quad \cdots \quad \mathbb{U}_{m1}^\top \quad \cdots \quad \mathbb{U}_{m\bar{n}}^\top \right]^\top, \quad (2.70)$$

$$\mathbb{V} = \left[ \mathbb{V}_{11}^\top \quad \cdots \quad \mathbb{V}_{1\bar{n}}^\top \quad \cdots \quad \mathbb{V}_{m1}^\top \quad \cdots \quad \mathbb{V}_{m\bar{n}}^\top \right]^\top, \quad (2.71)$$

$$\mathbb{M} = \left[ \mathbb{M}_{11}^\top \quad \cdots \quad \mathbb{M}_{1\bar{p}}^\top \quad \cdots \quad \mathbb{M}_{r1}^\top \quad \cdots \quad \mathbb{M}_{r\bar{p}}^\top \right]^\top, \quad (2.72)$$

$$\mathbb{N} = \left[ \mathbb{N}_{11}^\top \quad \cdots \quad \mathbb{N}_{1\bar{p}}^\top \quad \cdots \quad \mathbb{N}_{r1}^\top \quad \cdots \quad \mathbb{N}_{r\bar{p}}^\top \right]^\top. \quad (2.73)$$

Further, the terms  $\mathbb{V}$  and  $\mathbb{N}$  can be expressed in the following manner:

$$\mathbb{V} = H \Phi, \quad \text{and} \quad \mathbb{N} = G \Psi,$$

where,

$$H = \text{block-diag}(\underbrace{H_1, \dots, H_1}_{\bar{n} \text{ times}}, \dots, \underbrace{H_m, \dots, H_m}_{\bar{n} \text{ times}}), \quad (2.74)$$

$$G = \text{block-diag}(\underbrace{G_1, \dots, G_1}_{\bar{p} \text{ times}}, \dots, \underbrace{G_r, \dots, G_r}_{\bar{p} \text{ times}}), \quad (2.75)$$

$$\Phi^\top = [f_{11}\mathbb{I} \quad \dots \quad f_{1\bar{n}}\mathbb{I} \quad \dots \quad f_{m1}\mathbb{I} \quad \dots \quad f_{m\bar{n}}\mathbb{I}], \quad (2.76)$$

$$\Psi^\top = [g_{11}\mathbb{I} \quad \dots \quad g_{1\bar{p}}\mathbb{I} \quad \dots \quad g_{r1}\mathbb{I} \quad \dots \quad g_{r\bar{p}}\mathbb{I}]. \quad (2.77)$$

Thus, (2.68) and (2.69) are modified as:

$$N_1 = \mathbb{U}^\top (H\Phi) + (H\Phi)^\top \mathbb{U}, \quad (2.78)$$

$$N_2 = \mathbb{M}^\top (G\Psi) + (G\Psi)^\top \mathbb{M}. \quad (2.79)$$

### 2.5.2 LMI design with Young inequality

In this part of the chapter, the enhanced LMI condition is developed by using (A.9) and more generalised matrix multipliers.

The ensuing inequalities are obtained by implementing Young inequality (A.9) on (2.78) and (2.79):

$$N_1 \leq \mathbb{U}^\top \mathbb{Z}^{-1} \mathbb{U} + \Phi^\top H^\top \mathbb{Z} H \Phi, \quad (2.80)$$

$$N_2 \leq \mathbb{M}^\top \mathbb{S}^{-1} \mathbb{M} + \Psi^\top G^\top \mathbb{S} G \Psi, \quad (2.81)$$

where matrices  $\mathbb{Z}$  and  $\mathbb{S}$  are described in (2.82) and (2.83), respectively.

Let us define

$$\Phi_m^\top = [f_{b_{11}}\mathbb{I} \quad \dots \quad f_{b_{1\bar{n}}}\mathbb{I} \quad \dots \quad f_{b_{m1}}\mathbb{I} \quad \dots \quad f_{b_{m\bar{n}}}\mathbb{I}], \quad (2.84)$$

and

$$\Psi_m^\top = [g_{b_{11}}\mathbb{I} \quad \dots \quad g_{b_{1\bar{p}}}\mathbb{I} \quad \dots \quad g_{b_{r1}}\mathbb{I} \quad \dots \quad g_{b_{r\bar{p}}}\mathbb{I}]. \quad (2.85)$$

As  $\mathbb{Z} > 0$  and  $\mathbb{S} > 0$ , we get  $H^\top \mathbb{Z} H \geq 0$  and  $G^\top \mathbb{S} G \geq 0$ . Since each element of  $\Phi$  and  $\Phi_m$  fulfils (2.42), one can obtain the successive inequalities through the utilisation of Corollary 2.2:

$$\Phi^\top H^\top \mathbb{Z} H \Phi \leq \Phi_m^\top H^\top \mathbb{Z} H \Phi_m, \quad (2.86)$$

$$\Psi^\top G^\top \mathbb{S} G \Psi \leq \Psi_m^\top G^\top \mathbb{S} G \Psi_m. \quad (2.87)$$

It leads to

$$N_1 \leq \mathbb{U}^\top \mathbb{Z}^{-1} \mathbb{U} + \Phi_m^\top H^\top \mathbb{Z} H \Phi_m, \quad (2.88)$$

$$N_2 \leq \mathbb{M}^\top \mathbb{S}^{-1} \mathbb{M} + \Psi_m^\top G^\top \mathbb{S} G \Psi_m. \quad (2.89)$$

Therefore, condition (2.67) is satisfied if

$$\mathbb{A}(A, P, R) + \mathbb{U}^\top \mathbb{Z}^{-1} \mathbb{U} + \Phi_m^\top H^\top \mathbb{Z} H \Phi_m + \mathbb{M}^\top \mathbb{S}^{-1} \mathbb{M} + \Psi_m^\top G^\top \mathbb{S} G \Psi_m \leq 0. \quad (2.90)$$

With the necessary background established, the forthcoming theorem can be articulated, encapsulating a new LMI condition.

$$\mathbb{Z} = \begin{bmatrix} Z_{11} & Z_{a_{12}^1} & \dots & Z_{a_{1\bar{n}}^1} & Z_{b_{21}^{11}} & Z_{b_{22}^{11}} & \dots & Z_{b_{2\bar{n}}^{11}} & \dots & Z_{b_{m1}^{11}} & Z_{b_{m2}^{11}} & \dots & Z_{b_{m\bar{n}}^{11}} \\ Z_{a_{12}^1} & Z_{12} & \dots & Z_{a_{1\bar{n}}^2} & Z_{b_{21}^{12}} & Z_{b_{22}^{12}} & \dots & Z_{b_{2\bar{n}}^{12}} & \dots & Z_{b_{m1}^{12}} & Z_{b_{m2}^{12}} & \dots & Z_{b_{m\bar{n}}^{12}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ Z_{a_{1\bar{n}}^1} & Z_{a_{1\bar{n}}^2} & \dots & Z_{1\bar{n}} & Z_{b_{21}^{1\bar{n}}} & Z_{b_{22}^{1\bar{n}}} & \dots & Z_{b_{2\bar{n}}^{1\bar{n}}} & \dots & Z_{b_{m1}^{1\bar{n}}} & Z_{b_{m2}^{1\bar{n}}} & \dots & Z_{b_{m\bar{n}}^{1\bar{n}}} \\ Z_{b_{21}^{11}} & Z_{b_{21}^{12}} & \dots & Z_{b_{21}^{1\bar{n}}} & Z_{21} & Z_{a_{22}^1} & \dots & Z_{a_{2\bar{n}}^1} & \dots & Z_{b_{m1}^{21}} & Z_{b_{m2}^{21}} & \dots & Z_{b_{m\bar{n}}^{21}} \\ Z_{b_{22}^{11}} & Z_{b_{22}^{12}} & \dots & Z_{b_{22}^{1\bar{n}}} & Z_{a_{22}^1} & Z_{22} & \dots & Z_{a_{2\bar{n}}^2} & \dots & Z_{b_{m1}^{22}} & Z_{b_{m2}^{22}} & \dots & Z_{b_{m\bar{n}}^{22}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ Z_{b_{2\bar{n}}^{11}} & Z_{b_{2\bar{n}}^{12}} & \dots & Z_{b_{2\bar{n}}^{1\bar{n}}} & Z_{a_{2\bar{n}}^1} & Z_{a_{2\bar{n}}^2} & \dots & Z_{2\bar{n}} & \dots & Z_{b_{m1}^{2\bar{n}}} & Z_{b_{m2}^{2\bar{n}}} & \dots & Z_{b_{m\bar{n}}^{2\bar{n}}} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ Z_{b_{m1}^{11}} & Z_{b_{m1}^{12}} & \dots & Z_{b_{m1}^{1\bar{n}}} & Z_{b_{m1}^{21}} & Z_{b_{m1}^{22}} & \dots & Z_{b_{m1}^{2\bar{n}}} & \dots & Z_{m1} & Z_{a_{m2}^1} & \dots & Z_{a_{m\bar{n}}^1} \\ Z_{b_{m2}^{11}} & Z_{b_{m2}^{12}} & \dots & Z_{b_{m2}^{1\bar{n}}} & Z_{b_{m2}^{21}} & Z_{b_{m2}^{22}} & \dots & Z_{b_{m2}^{2\bar{n}}} & \dots & Z_{a_{m2}^1} & Z_{m2} & \dots & Z_{a_{m\bar{n}}^2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ Z_{b_{m\bar{n}}^{11}} & Z_{b_{m\bar{n}}^{12}} & \dots & Z_{b_{m\bar{n}}^{1\bar{n}}} & Z_{b_{m\bar{n}}^{21}} & Z_{b_{m\bar{n}}^{22}} & \dots & Z_{b_{m\bar{n}}^{2\bar{n}}} & \dots & Z_{a_{m\bar{n}}^1} & Z_{a_{m\bar{n}}^2} & \dots & Z_{m\bar{n}} \end{bmatrix}, \quad (2.82)$$

where  $Z_{ij} > 0 \in \mathbf{S}^{\bar{n}}$ ,  $Z_{a_{ij}^k} \geq 0 \in \mathbf{S}^{\bar{n}} \forall i, k \in \{1, \dots, m\}, \&j \in \{1, \dots, \bar{n}\}$ ;  $Z_{b_{ij}^{kj}} \geq 0 \in \mathbf{S}^{\bar{n}}, \forall i \in \{2, \dots, m\}, k \in \{1, \dots, m-1\}, \&j \in \{1, \dots, \bar{n}\}$  such that  $\mathbb{Z} > 0$ .

$$\mathbb{S} = \begin{bmatrix} S_{11} & S_{a_{12}^1} & \dots & S_{a_{1\bar{p}}^1} & S_{b_{21}^{11}} & S_{b_{22}^{11}} & \dots & S_{b_{2\bar{p}}^{11}} & \dots & S_{b_{r1}^{11}} & S_{b_{r2}^{11}} & \dots & S_{b_{r\bar{p}}^{11}} \\ S_{a_{12}^1} & S_{12} & \dots & S_{a_{1\bar{p}}^2} & S_{b_{21}^{12}} & S_{b_{22}^{12}} & \dots & S_{b_{2\bar{p}}^{12}} & \dots & S_{b_{r1}^{12}} & S_{b_{r2}^{12}} & \dots & S_{b_{r\bar{p}}^{12}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ S_{a_{1\bar{p}}^1} & S_{a_{1\bar{p}}^2} & \dots & S_{1\bar{p}} & S_{b_{21}^{1\bar{p}}} & S_{b_{22}^{1\bar{p}}} & \dots & S_{b_{2\bar{p}}^{1\bar{p}}} & \dots & S_{b_{r1}^{1\bar{p}}} & S_{b_{r2}^{1\bar{p}}} & \dots & S_{b_{r\bar{p}}^{1\bar{p}}} \\ S_{b_{21}^{11}} & S_{b_{21}^{12}} & \dots & S_{b_{21}^{1\bar{p}}} & S_{21} & S_{a_{22}^1} & \dots & S_{a_{2\bar{p}}^1} & \dots & S_{b_{r1}^{21}} & S_{b_{r2}^{21}} & \dots & S_{b_{r\bar{p}}^{21}} \\ S_{b_{22}^{11}} & S_{b_{22}^{12}} & \dots & S_{b_{22}^{1\bar{p}}} & S_{a_{22}^1} & S_{22} & \dots & S_{a_{2\bar{p}}^2} & \dots & S_{b_{r1}^{22}} & S_{b_{r2}^{22}} & \dots & S_{b_{r\bar{p}}^{22}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ S_{b_{2\bar{p}}^{11}} & S_{b_{2\bar{p}}^{12}} & \dots & S_{b_{2\bar{p}}^{1\bar{p}}} & S_{a_{2\bar{p}}^1} & S_{a_{2\bar{p}}^2} & \dots & S_{2\bar{p}} & \dots & S_{b_{r1}^{2\bar{p}}} & S_{b_{r2}^{2\bar{p}}} & \dots & S_{b_{r\bar{p}}^{2\bar{p}}} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ S_{b_{r1}^{11}} & S_{b_{r1}^{12}} & \dots & S_{b_{r1}^{1\bar{p}}} & S_{b_{r1}^{21}} & S_{b_{r1}^{22}} & \dots & S_{b_{r1}^{2\bar{p}}} & \dots & S_{r1} & S_{a_{r2}^1} & \dots & S_{a_{r\bar{p}}^1} \\ S_{b_{r2}^{11}} & S_{b_{r2}^{12}} & \dots & S_{b_{r2}^{1\bar{p}}} & S_{b_{r2}^{21}} & S_{b_{r2}^{22}} & \dots & S_{b_{r2}^{2\bar{p}}} & \dots & S_{a_{r2}^1} & S_{r2} & \dots & S_{a_{r\bar{p}}^2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ S_{b_{r\bar{p}}^{11}} & S_{b_{r\bar{p}}^{12}} & \dots & S_{b_{r\bar{p}}^{1\bar{p}}} & S_{b_{r\bar{p}}^{21}} & S_{b_{r\bar{p}}^{22}} & \dots & S_{b_{r\bar{p}}^{2\bar{p}}} & \dots & S_{a_{r\bar{p}}^1} & S_{a_{r\bar{p}}^2} & \dots & S_{r\bar{p}} \end{bmatrix}, \quad (2.83)$$

where  $S_{ij} > 0 \in \mathbf{S}^{\bar{p}}$ ,  $S_{a_{ij}^k} \geq 0 \in \mathbf{S}^{\bar{p}} \forall i, k \in \{1, \dots, r\}, \&j \in \{1, \dots, \bar{p}\}$ ;  $S_{b_{ij}^{kj}} \geq 0 \in \mathbf{S}^{\bar{p}}, \forall i \in \{2, \dots, r\}, k \in \{1, \dots, r-1\}, \&j \in \{1, \dots, \bar{p}\}$  so that  $\mathbb{S} > 0$ .

#### Theorem 2.4

Let us assume that there exist the matrices  $\mathbb{Z}$  and  $\mathbb{S}$  as specified in (2.82) and (2.83), respectively, along with matrices  $P > 0 \in \mathbf{S}^n$  and  $R \in \mathbb{R}^{p \times n}$ , and a positive scalar  $\epsilon$ . The estimation error dynamic (2.65) fulfils the exponential stability criterion (2.45) if the following LMI is

feasible:

$$\begin{bmatrix} \mathbb{A}(A, P, R) & \mathbb{U}^\top & \Phi_m^\top H^\top \mathbb{Z} & \mathbb{M}^\top & \Psi_m^\top G^\top \mathbb{S} \\ \star & -\mathbb{Z} & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \star & \star & -\mathbb{Z} & \mathbb{O} & \mathbb{O} \\ \star & \star & \star & -\mathbb{S} & \mathbb{O} \\ \star & \star & \star & \star & -\mathbb{S} \end{bmatrix} < 0, \quad (2.91)$$

where matrices  $\mathbb{U}$ ,  $\mathbb{M}$ ,  $H$ ,  $G$ ,  $\Phi_m$  and  $\Psi_m$  are illustrated in (2.70), (2.72), (2.74), (2.75), (2.84) and (2.85) respectively. Other variables and parameters are defined in Theorem 2.3. The gain of the observer is calculated by using  $L = P^{-1}R^\top$ .

**Proof:** The proof of the theorem is obvious, as the Schur complement of (2.90) yields an LMI (2.91), which ensures exponential stability of the error dynamics (2.65).  $\square$

### 2.5.3 LPV-based LMI approach

Recently, the new variant of Young relation (A.10) is used in the LMI-based observer design to derive an enhanced LMI condition (for example, [14] and [22]). In this subsection, a well-known LPV approach is incorporated to deduce a novel LMI condition.

The subsequent inequalities are derived by applying inequality (A.10) on (2.78) and (2.79):

$$N_1 \leq \frac{1}{2}(\mathbb{U} + \mathbb{Z}H\Phi)^\top \mathbb{Z}^{-1}(\mathbb{U} + \mathbb{Z}H\Phi), \quad (2.92)$$

$$N_2 \leq \frac{1}{2}(\mathbb{M} + \mathbb{S}G\Psi)^\top \mathbb{S}^{-1}(\mathbb{M} + \mathbb{S}G\Psi), \quad (2.93)$$

where matrices  $\mathbb{Z}$  and  $\mathbb{S}$  are defined in (2.82) and (2.83), respectively.

Therefore, inequality (2.67) holds if

$$\mathbb{A}(A, P, R) + \frac{1}{2} \left[ (\mathbb{U} + \mathbb{Z}H\Phi)^\top \mathbb{Z}^{-1}(\mathbb{U} + \mathbb{Z}H\Phi) \right] + \frac{1}{2} \left[ (\mathbb{M} + \mathbb{S}G\Psi)^\top \mathbb{S}^{-1}(\mathbb{M} + \mathbb{S}G\Psi) \right] \leq 0. \quad (2.94)$$

From (2.42), and (2.64), each element inside  $\Phi$  and  $\Psi$  are bounded and belong to convex sets  $\mathcal{F}_m$  and  $\mathcal{G}_r$ , respectively.

The sets  $\mathcal{F}_m$  and  $\mathcal{G}_r$  are defined as follows:

$$\begin{aligned} \mathcal{F}_m &\triangleq \{ \Phi : 0 \leq f_{ij} \leq f_{b_{ij}}, \forall i \in \{1, \dots, m\} \& j \in \{1, \dots, \bar{n}\} \}, \\ \mathcal{G}_r &\triangleq \{ \Psi : 0 \leq g_{ij} \leq g_{b_{ij}}, \forall i \in \{1, \dots, r\} \& j \in \{1, \dots, \bar{p}\} \}. \end{aligned}$$

The set of vertices of  $\mathcal{F}_m$  and  $\mathcal{G}_r$  are given by

$$\mathcal{F}_{H_m} = \left\{ \{ \mathcal{F}_{11}, \dots, \mathcal{F}_{1\bar{n}}, \dots, \mathcal{F}_{m1}, \dots, \mathcal{F}_{m\bar{n}} \} : \mathcal{F}_{ij} \in [0, f_{b_{ij}}] \right\}, \quad (2.95)$$

$$\mathcal{G}_{G_r} = \left\{ \{ \mathcal{G}_{11}, \dots, \mathcal{G}_{1\bar{p}}, \dots, \mathcal{G}_{r1}, \dots, \mathcal{G}_{r\bar{p}} \} : \mathcal{G}_{ij} \in [0, g_{b_{ij}}] \right\}. \quad (2.96)$$

Hence, (2.94) is rewritten as:

$$\begin{aligned} \mathbb{A}(A, P, R) + \left[ \frac{1}{2}(\mathbb{U} + \mathbb{Z}H\Phi)^\top \mathbb{Z}^{-1}(\mathbb{U} + \mathbb{Z}H\Phi) \right]_{\forall \Phi \in \mathcal{F}_{H_m}} \\ + \left[ \frac{1}{2}(\mathbb{M} + \mathbb{S}G\Psi)^\top \mathbb{S}^{-1}(\mathbb{M} + \mathbb{S}G\Psi) \right]_{\forall \Psi \in \mathcal{G}_{G_r}} \leq 0. \end{aligned} \quad (2.97)$$

After establishing the fundamental framework, we can define the following theorem.

**Theorem 2.5**

Let us consider the matrices  $\mathbb{Z}$  and  $\mathbb{S}$  as specified in (2.82) and (2.83), respectively, along with matrices  $P > 0 \in \mathbf{S}^n$  and  $R \in \mathbb{R}^{p \times n}$  and a positive scalar  $\epsilon$ . The estimation error dynamic (2.65) achieves exponential stability if the following LMI is feasible.

$$\begin{bmatrix} \mathbb{A}(A, P, R) & (\mathbb{U} + \mathbb{Z}H\Phi)^\top & (\mathbb{M} + \mathbb{S}G\Psi)^\top \\ \star & -2\mathbb{Z} & \mathbb{O} \\ \star & \star & -2\mathbb{S} \end{bmatrix} < 0, \forall \Phi \in \mathcal{F}_{F_m}, \forall \Psi \in \mathcal{G}_{G_r}, \quad (2.98)$$

where all variables and parameters remain consistent with those specified in Theorem 2.4 .

**Proof:** The LMI (2.98) is derived by implementing Schur lemma on (2.97). From convexity principal [18], the exponential stability criterion (2.45) is satisfied by estimation error dynamics (2.65) if LMI (2.98) is solved for all  $\Phi \in \mathcal{F}_{F_m}$  and  $\Psi \in \mathcal{G}_{G_r}$ .  $\square$

**2.5.4 LMIs for the nonlinear systems with linear outputs**

The proposed LMIs (2.91) and (2.98) can be reconfigured for the system (2.33).

The subsequent corollaries present the novel LMI conditions for the computation of the observer parameter of (2.35).

**Corollary 2.3**

The estimation error dynamic (2.43) fulfils the exponential stability criterion (2.45) if the following LMI is feasible:

$$\begin{bmatrix} \mathbb{A}(A, P, R) & \mathbb{U}^\top & \Phi_m^\top H^\top \mathbb{Z} \\ \star & -\mathbb{Z} & \mathbb{O} \\ \star & \star & -\mathbb{Z} \end{bmatrix} < 0, \quad (2.99)$$

where all variables and parameters remain coherent with those specified in Theorem 2.4 .

Similarly,

**Corollary 2.4**

The feasibility of the following LMI indicates that the estimation error dynamic (2.43) exhibits exponential stability, i.e., it satisfies criterion (2.45).

$$\begin{bmatrix} \mathbb{A}(A, P, R) & (\mathbb{U} + \mathbb{Z}H\Phi)^\top \\ \star & -2\mathbb{Z} \end{bmatrix} < 0, \forall \Phi \in \mathcal{F}_{F_m}, \quad (2.100)$$

where all variables and parameters are the same as the ones described in Theorem 2.5 .

The proof of both corollaries is straightforward. If one follows the proof of Theorem 2.4 and Theorem 2.5 , it is easy to prove the corollaries mentioned above.

## 2.6 Validation of proposed methodologies

This section is dedicated to highlighting the validity and effectiveness of the proposed methodology through numerical examples and practical application. First, an example of a nonlinear system with linear outputs is considered to analyse the effect of various parameters on the LMI feasibility. Further, another numerical example is used to compare the efficiency of the proposed LMIs with the existing LMI methods. In addition to this, the practical application of a flexible link robot model described in [11] is used to demonstrate the performance of the observer (2.35).

### 2.6.1 Numerical example 1

The following numerical example is taken from [14], and it is used to analyse the effect of the parameter  $\alpha$  on the feasibility of LMI (2.55).

Consider the nonlinear system described in (2.33) with following parameters:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, G = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}.$$

The nonlinear function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a differential function. Without loss of generality, we assume that  $f$  satisfies condition (2.42), i.e.,

$$0 \leq \frac{\partial f}{\partial \nu_j} \leq \theta, \text{ for } j = 1, 2. \quad (2.101)$$

Let us consider the following two cases of  $\bar{n}$ ,

$$\begin{aligned} \text{Case 1 : } H_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; & \bar{n} &= 2 \\ \text{Case 2 : } H_1 &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; & \bar{n} &= 3 \end{aligned}$$

Further, by using the YALMIP tool, LMI (2.55) is solved for this example and the LMI feasibility is analysed for the following two cases of  $\bar{n}$ .

Table 2.4: Maximum value of  $\theta$  (i.e.,  $\theta_m$ ) for which LMI (2.55) is feasible

		$\alpha = 1$	$\alpha = 0.75$	$\alpha = 0.5$	$\alpha = 0.25$	$\alpha = 0.1$
Case 1	$\theta_m$	0.249	0.285	0.333	0.399	0.454
Case 2	$\theta_m$	0.333	0.399	0.499	0.65	0.747

The effect of  $\alpha$  on the feasibility of (2.55) is clearly seen in Tables 2.4. The term  $\theta_m$  indicates the maximum value of  $\theta$  for which LMI (2.55) gives the feasible solution. From these two tables,  $\theta_m$  is increasing as  $\alpha$  is reducing. Thus, LMI (2.55) has a larger feasibility region for smaller values of  $\alpha$ . The feasibility of a new matrix multiplier-based LMI (i.e., LMI (2.55)) is totally dependent on the parameters  $\alpha$  and  $\bar{n}$ , as seen in this example.

In the sequel, another numerical example is considered to analyse the effectiveness of LMIs (2.99) and (2.100).

### 2.6.2 Numerical example 2

Let us analyse the feasibility of the proposed LMIs for the system (2.33) with respect to the following parameters:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}; G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The nonlinearities present in this system are as follows:

1.  $f_1(H_1) = \sin(\theta x_1)\sin(\theta x_3)$  and  $H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ . Hence,

$$-\theta \leq \frac{\partial f_1}{\partial x_1} \leq \theta \text{ and } -\theta \leq \frac{\partial f_1}{\partial x_3} \leq \theta.$$

2.  $f_2(H_2x)$  is a function such that, it holds  $0 \leq \frac{\partial f_2}{\partial x_2} \leq \theta$ , along with  $H_2 = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ .

Hence,  $m = 2$  and  $\bar{n} = 3$ .

To highlight the effectiveness of the proposed LMIs, the following cases are considered.

1. **Case 1:** LMI (2.99) with

$$Z = \begin{bmatrix} Z_{11} & \alpha Z_{12} & \alpha Z_{21} \\ \star & Z_{12} & \alpha Z_{21} \\ \star & \star & Z_{21} \end{bmatrix}, \quad (2.102)$$

where  $Z_{11}, Z_{12}, Z_{21} \in \mathbb{R}^{\bar{n} \times \bar{n}}$  are symmetric positive definite matrices, and  $\alpha = 0.5$ .

2. **Case 2:** LMI (2.99) with

$$Z = \begin{bmatrix} Z_{11} & Z_{a_{12}} & Z_{b_{21}^1} \\ \star & Z_{12} & Z_{b_{21}^2} \\ \star & \star & Z_{21} \end{bmatrix}, \quad (2.103)$$

where  $Z_{11}, Z_{12}, Z_{21} \in \mathbb{R}^{\bar{n} \times \bar{n}}$  are symmetric positive definite matrices, and  $Z_{a_{12}}, Z_{b_{21}^1}, Z_{b_{21}^2} \in \mathbb{R}^{\bar{n} \times \bar{n}}$  are symmetric positive semi-definite matrices.

3. **Case 3:** LMI (2.100) along with a matrix  $\mathbb{Z}$  which is described in (2.103).

4. **Case 4:** LMI proposed in [21]

For different  $\theta$  values, the LMIs from the above cases are solved using the MATLAB toolbox. Table 2.5 indicates the maximum value of  $\theta$  (i.e.,  $\theta_m$ ) for which different LMIs are feasible. It highlights that the proposed LMI (2.100) provides a better solution. In other words, it has the maximum feasible region than the other methods described in the above cases. Thus, it validates the efficiency of the proposed LMI.

From the above analysis, one can interpret that the inclusion of the newly defined matrix multipliers aids in the enhancement of the LMI approach from a feasibility point of view. In the next chapter, the relaxation of the LMI conditions through the inclusion of matrix multipliers will be discussed in detail.

Table 2.5: Comparison of different LMI methods

	Case 1	Case 2	Case 3	Case 4
$\theta_m$	0.995	$6.528 \times 10^4$	$2.004 \times 10^6$	$1.413 \times 10^5$

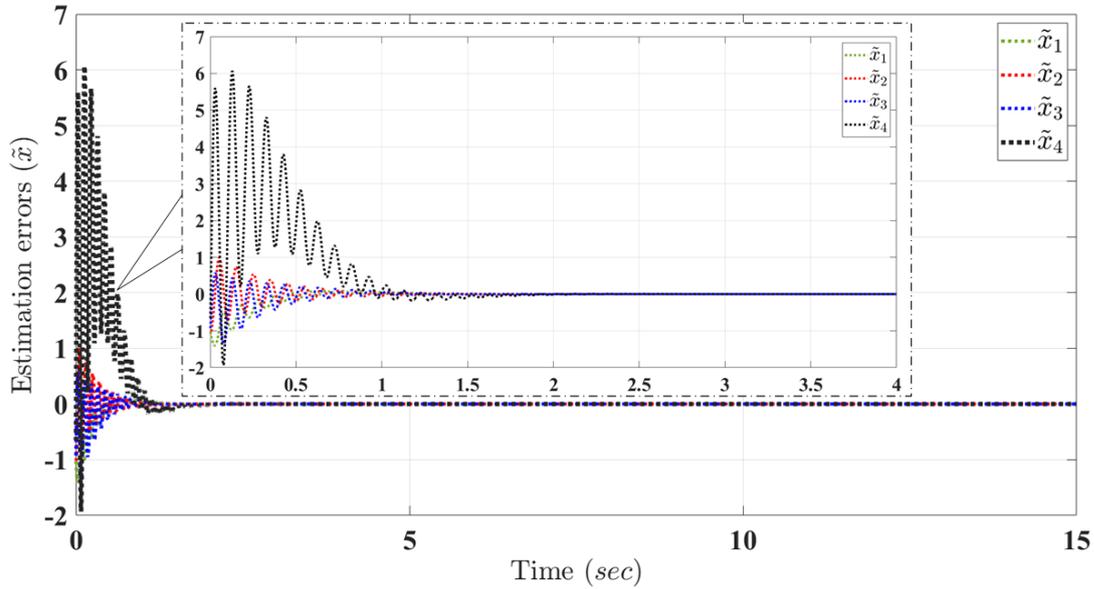


Figure 2.2: Plot of estimation error ( $\tilde{x}$ ) in **Case 1**

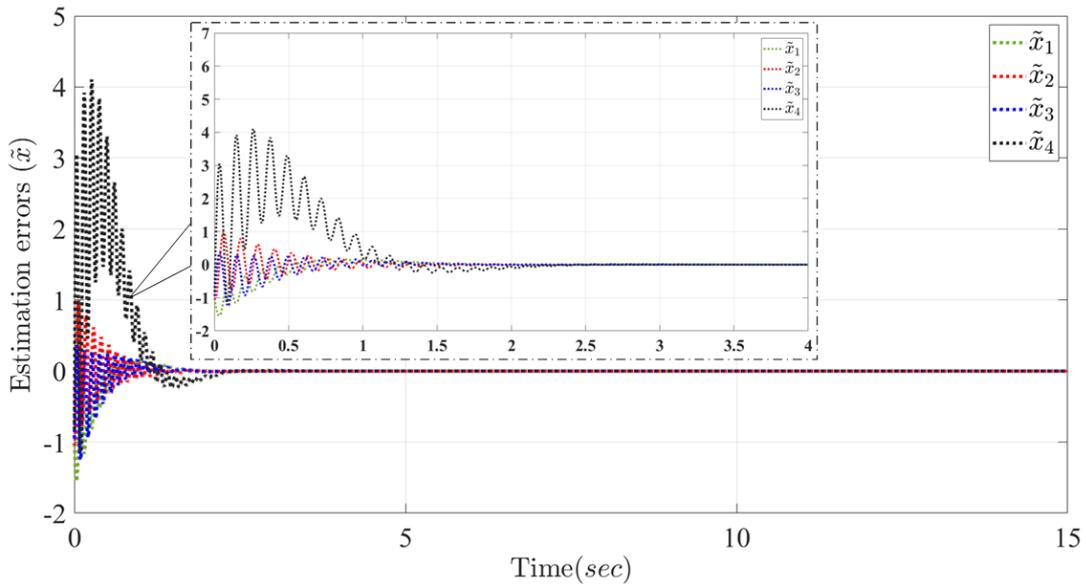
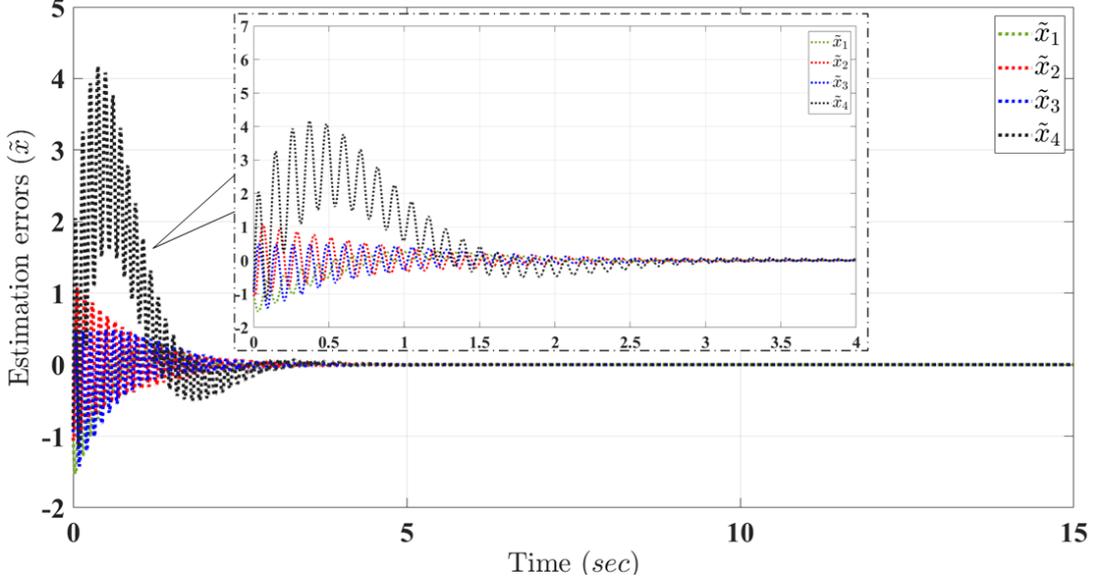


Figure 2.3: Plot of estimation error ( $\tilde{x}$ ) in **Case 2**

### 2.6.3 Implementation on a robotic arm manipulator

In this segment of the chapter, the observer performance is validated through a practical application, a flexible link robot model. The proposed observer design is implemented in MATLAB, and the effectiveness of the observer is evaluated.

Figure 2.4: Plot of estimation error ( $\tilde{x}$ ) in **Case 3**

Consider the system dynamics of the flexible link robot represented in the form of (2.33) with the following parameters,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix}, G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3.33 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Further,  $f(H_1x) = \sin(x_3)$ , where  $H_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ . Thus,  $m = 1$  and  $\bar{n} = 4$ .

The input of the system is assumed as  $u = \sin(t)$ . The initial states of system and observer are considered as  $x(0) = [1 \ 1 \ 1 \ 1]$  and  $\hat{x}(0) = [0 \ 0 \ 0 \ 0]$ , respectively.

The proposed LMIs, i.e., (2.55), (2.99) and (2.100) are solved in the MATLAB toolbox, and the obtained result is summarized in Table 2.6. Further, the observer (2.35) is implemented in MATLAB for the purpose of state estimation of robotic arm manipulator for each gain matrices.

Table 2.6: Representation of the results obtained from the solution of the proposed LMIs

Case 1	Case 2	Case 3
LMI (2.55) with $\alpha = 0.5$	LMI (2.99)	LMI (2.100)
$L = \begin{bmatrix} 1.5389 & -23.9559 \\ -22.8269 & 10.1797 \\ 5.6981 & 69.0578 \\ 53.7942 & 288.2865 \end{bmatrix}$	$L = \begin{bmatrix} 1.1694 & -24.3703 \\ -23.8745 & 6.6162 \\ 2.7935 & 48.9559 \\ 30.1371 & 142.4056 \end{bmatrix}$	$L = \begin{bmatrix} 0.7399 & -21.4370 \\ -25.3612 & 3.4518 \\ 2.7557 & 52.6295 \\ 26.9135 & 103.4519 \end{bmatrix}$

Figures 2.2, 2.3 and 2.4 depict the plot of estimation error  $\tilde{x}$  of nonlinear observer based on

the proposed LMIs (2.55), (2.99) and (2.100), respectively. In all cases, the estimation error is converging toward zero exponentially which highlights the effectiveness of a new matrix multiplier-based LMIs.

## 2.7 Conclusion

The agenda of this chapter was to introduce a new LMI-based observer design. In the context of this objective, a few novel approaches have been proposed in this chapter and validated through numerical examples. The key points of this chapter are listed as follows:

- 1 The use of reformulated Lipschitz property, i.e., Definition 3 of Appendix A.1 in LMI development to handle the Lipschitz nonlinearities;
- 2 The introduction of newly defined matrix multipliers, i.e., (2.82) and (2.83);
- 3 The judicious utilisation of Young inequalities (A.9) and (A.10) to derive new LMI conditions;
- 4 The inclusion of LPV approach to enhance the proposed LMI methodology;

All these major contributions lead to the introduction of additional numbers of decision variables inside LMIs and enrich the degrees of freedom. Hence, the proposed LMIs are more enhanced than the existing ones from a feasibility point.

---

# Enhancing the observers for the disturbance-affected nonlinear systems: a matrix multipliers-based LMI approach

---

## Contents

<b>3.1</b>	<b>Introduction</b>	<b>60</b>
<b>3.2</b>	<b>Design of <math>\mathcal{H}_\infty</math> nonlinear observer</b>	<b>61</b>
3.2.1	Articulating the problem statement	61
3.2.2	LMI formulation with Young standard inequality	65
3.2.3	An Enhanced LMI criterion: exploiting LPV approach	67
3.2.4	$\mathcal{H}_\infty$ observers for the nonlinear systems with linear outputs	68
<b>3.3</b>	<b>LMI-based observers for locally Lipschitz systems: exploiting an ISS framework</b>	<b>70</b>
3.3.1	Insight on the Hilbert projection	70
3.3.2	ISS-based observer design: articulating the problem statement	72
3.3.3	LMI synthesis	75
3.3.4	Observer design for systems having nonlinear dynamics and linear outputs	80
<b>3.4</b>	<b>Analysing the effectiveness of matrix-multipliers-based LMIs</b>	<b>81</b>
3.4.1	Number of decision variables	81
3.4.2	A comparison of the number of additional decision variables in different matrix multiplier structures	82
<b>3.5</b>	<b>Evaluation of the proposed LMIs and the observers</b>	<b>85</b>
3.5.1	Numerical Example 1	85
3.5.2	Numerical Example 2	89
<b>3.6</b>	<b>Conclusion</b>	<b>92</b>

---

### 3.1 Introduction

In the previous chapter, novel nonlinear observer design methods are developed for state estimation purposes. However, most of the system dynamics and outputs are frequently corrupted by exogenous signals such as noise/disturbances. Therefore, it is essential to consider the impact of such external factors during the state estimation process in order to achieve precise state reconstructions by the proposed observers. In the literature, an abundant amount of research is conducted on the development of observers for these types of systems, for instance, the authors of [68] have developed an LMI-based nonlinear observer to estimate the state-of-charge (SoC) of the Li-ion battery model. In [13, 17], an  $\mathcal{H}_\infty$  observer is designed to estimate the states of the neural mass model under the presence of disturbances/noise. Furthermore, an observer for the control of glucose levels in type-2 diabetic patients is proposed in [69, 70]. Therefore, it motivates the author to deploy the proposed matrix-multiplier-based LMI approach in the problem of observer design for the disturbance-affected systems.

The objective of this chapter is to design an observer using a novel LMI condition for the nonlinear system whose dynamics and outputs are corrupted by disturbance/noise. The primary component of the novel LMIs is the newly defined matrix multipliers. The deployment of such a matrix multiplier adds some additional numbers of decision variables inside LMIs and enhances the LMI feasibility.

The rest of the chapter is structured as follows: Section 3.2 encompasses the design of  $\mathcal{H}_\infty$  nonlinear observer. The development of the LMI condition based on standard Young inequality is illustrated in Subsection 3.2.1, whereas Subsection 3.2.2 includes the formulation of the LPV-based LMI approach. The extension of the proposed new matrix-multiplier based-LMI approach for the nonlinear system with local Lipschitz nonlinearities is showcased in Section 3.3. Further, Section 3.4 emphasises the effect of the different structures of matrix multipliers on the proposed LMIs. The enhancement of the developed LMI conditions and the validation of the proposed observer are demonstrated in section 3.5 through numerical examples. Some conclusions are drawn in Section 3.6.

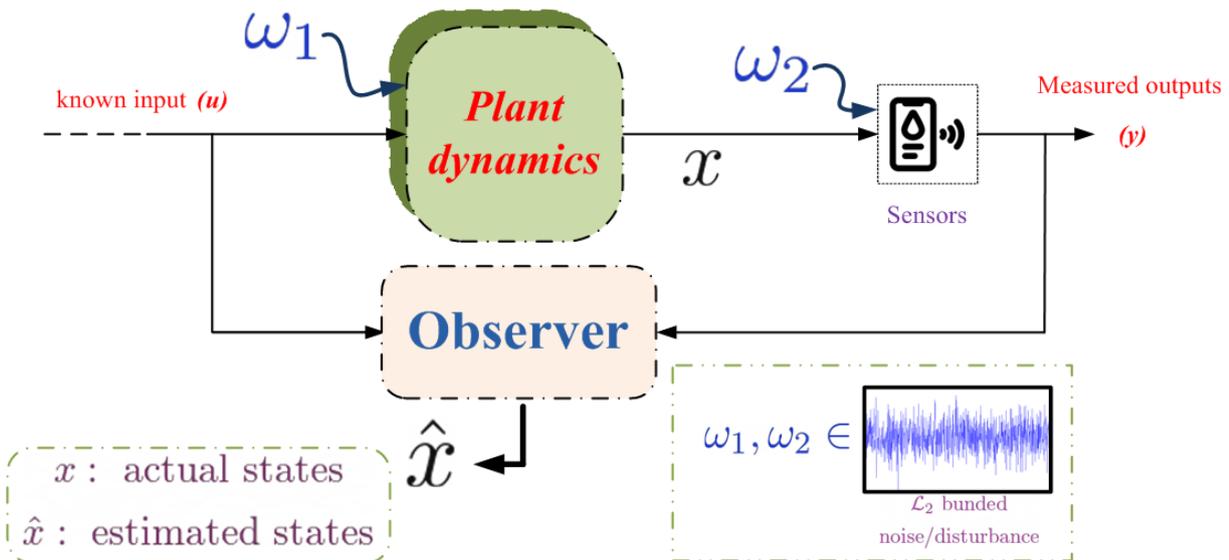


Figure 3.1: Visualisation of disturbance-affected nonlinear system

## 3.2 Design of $\mathcal{H}_\infty$ nonlinear observer

This section is dedicated to the observer design for a disturbance-affected nonlinear system.

### 3.2.1 Articulating the problem statement

Let us consider that a class of nonlinear systems with nonlinear outputs in the presence of exogenous signals such as noise or disturbances is expressed by the subsequent equations:

$$\begin{aligned} \dot{x} &= Ax + Gf(x) + Bu + E_1\omega_1, \\ y &= Cx + Fg(x) + D_1\omega_2, \end{aligned} \quad (3.1)$$

where

- i)  $x \in \mathbb{R}^n$  and  $y \in \mathbb{R}^p$  denote the states and the measurements of the systems, respectively.
- ii)  $u \in \mathbb{R}^s$  depicts the input of the system.
- iii)  $A, G, B, C, F$  are known constant matrices of appropriate dimensions.
- iv)  $\omega_1 \in \mathbb{R}^{q_1}$  and  $\omega_2 \in \mathbb{R}^{q_2}$  are the exogenous signals such as noise or disturbances affecting the system dynamics and outputs, respectively. Matrices  $E_1 \in \mathbb{R}^{n \times q_1}$  and  $D_1 \in \mathbb{R}^{p \times q_2}$  are known and constant.

The graphical representation of system (3.1) is shown in Figure 3.1.

For the enhancement of comprehensibility, the system (3.1) is written under the ensuing form:

$$\begin{aligned} \dot{x} &= Ax + Gf(x) + Bu + E\omega, \\ y &= Cx + Fg(x) + D\omega, \end{aligned} \quad (3.2)$$

where

$$E = \begin{bmatrix} E_1 & 0 \end{bmatrix}; \quad D = \begin{bmatrix} 0 & D_1 \end{bmatrix} \quad \text{and} \quad \omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}.$$

The system (3.1) represents a more generalised form and is often encountered in practical situations. Whereas, the form specified in (3.2) facilitates the simplification of the observer design. Therefore, the nonlinear systems (3.2) is considered for the rest of this section.

#### Remark 3.1

Since there is no assumption regarding the structure of the matrices  $E_1, D_1$ , or the dimension of the disturbances  $\omega_1$  and  $\omega_2$ , one can easily obtain the model (3.2) from (3.1).

The functions  $f(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $g(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^r$  are assumed to be globally Lipschitz. The detailed form of functions  $f(\cdot)$  and  $g(\cdot)$  are described in (2.34) and (2.58), respectively.

The following Luenberger observer is deployed for the state estimation of the system (3.2):

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Gf(\hat{x}) + Bu + L(y - \hat{y}), \\ \hat{y} &= C\hat{x} + Fg(\hat{x}), \end{aligned} \quad (3.3)$$

where  $\hat{x}$ ,  $\hat{y}$  denote estimated states and observer outputs, respectively, and  $L \in \mathbb{R}^{n \times p}$  is the observer gain matrix.

The estimation error of the observer (3.3) is defined as

$$\tilde{x} = x - \hat{x}.$$

From (3.2) and (3.3), the dynamics of the estimation error ( $\tilde{x}$ ) is computed and illustrated as follows:

$$\dot{\tilde{x}} = \mathcal{A}\tilde{x} + G\tilde{f}(x, \hat{x}) - LF\tilde{g}(x, \hat{x}) + \mathbb{E}\omega, \quad (3.4)$$

where

$$\mathcal{A} = A - LC, \quad (3.5)$$

$$\mathbb{E} = E - LD, \quad (3.6)$$

$$\tilde{f}(x, \hat{x}) = f(x) - f(\hat{x}), \quad (3.7)$$

$$\tilde{g}(x, \hat{x}) = g(x) - g(\hat{x}). \quad (3.8)$$

The terms  $\tilde{f}(x, \hat{x})$  and  $\tilde{g}(x, \hat{x})$  are elaborated in (2.39) and (2.61), respectively. Further, the subsequent equations are computed by deploying Lemma 2.1 :

$$\tilde{f}(x, \hat{x}) = \sum_{i,j=1}^{m,\bar{n}} f_{ij} \mathcal{H}_{ij} H_i \tilde{x}, \quad (3.9)$$

and

$$\tilde{g}(x, \hat{x}) = \sum_{i,j=1}^{r,\bar{p}} g_{ij} \mathcal{G}_{ij} G_i \tilde{x}, \quad (3.10)$$

where functions  $f_{ij}$  and  $g_{ij}$  are defined in (2.40) and (2.62), respectively.  $\mathcal{H}_{ij} = e_m(i)e_{\bar{n}}^\top(j)$  and  $\mathcal{G}_{ij} = e_r(i)e_{\bar{p}}^\top(j)$ . Both functions are bounded as follows:

$$f_{a_{ij}} \leq f_{ij} \leq f_{b_{ij}}, \quad (3.11)$$

$$g_{a_{ij}} \leq g_{ij} \leq g_{b_{ij}}. \quad (3.12)$$

Without loss of generality, let us assume that  $f_{a_{ij}} = 0$  and  $g_{a_{ij}} = 0$ . Thus, conditions (3.11) and (3.12) are reformulated as follows:

$$0 \leq f_{ij} \leq f_{b_{ij}}, \quad (3.13)$$

$$0 \leq g_{ij} \leq g_{b_{ij}}. \quad (3.14)$$

By using (3.9) and (3.10), the error dynamic (3.4) is altered as:

$$\dot{\tilde{x}} = \mathcal{A}\tilde{x} + \sum_{i,j=1}^{m,\bar{n}} f_{ij} G \mathcal{H}_{ij} H_i \tilde{x} - \sum_{i,j=1}^{r,\bar{p}} g_{ij} L F \mathcal{G}_{ij} G_i \tilde{x} + \mathbb{E}\omega. \quad (3.15)$$

### Remark 3.2

In various practical applications, it is possible to have  $f_{a_{ij}}, g_{a_{ij}} \neq 0$ . In such cases, (3.15) is

rewritten as:

$$\begin{aligned} \dot{\tilde{x}} = & \underbrace{\left( \mathbb{A} + \sum_{i,j=1}^{m,\bar{n}} f_{a_{ij}} G \mathcal{H}_{ij} H_i - \sum_{i,j=1}^{r,\bar{p}} g_{a_{ij}} L F \mathcal{G}_{ij} G_i \right)}_{\tilde{\mathcal{A}}} \tilde{x} + \sum_{i,j=1}^{m,\bar{n}} \underbrace{(f_{ij} - f_{a_{ij}})}_{\tilde{f}_{ij}} G \mathcal{H}_{ij} H_i \tilde{x} \\ & - \sum_{i,j=1}^{r,\bar{p}} \underbrace{(g_{ij} - g_{a_{ij}})}_{\tilde{g}_{ij}} L F \mathcal{G}_{ij} G_i \tilde{x} + \mathbb{E} \omega. \end{aligned}$$

Hence,

$$\dot{\tilde{x}} = \tilde{\mathcal{A}} \tilde{x} + \sum_{i,j=1}^{m,\bar{n}} \tilde{f}_{ij} G \mathcal{H}_{ij} H_i \tilde{x} - \sum_{i,j=1}^{r,\bar{p}} \tilde{g}_{ij} L F \mathcal{G}_{ij} G_i \tilde{x} + \mathbb{E} \omega. \quad (3.16)$$

For the error dynamics (3.16), functions  $\tilde{f}_{ij}$  and  $\tilde{g}_{ij}$  fulfil (3.13) and (3.14), respectively. In addition to this, both forms, i.e., (3.16) and (3.15) are identical.

Now, we can formulate the problem statement for this segment of the chapter, which is illustrated as follows:

### Objective

The purpose of this section is to determine the observer gain  $L$  such that

- I) When  $\omega = 0$ , the estimation error dynamic (3.15) is converging towards zero at  $t \rightarrow \infty$ .
- II) When  $\omega \neq 0$ , the estimation error dynamic (3.15) satisfies the  $\mathcal{H}_\infty$  criterion:

$$\|\tilde{x}\|_{\mathcal{L}_2^n} \leq \sqrt{\mu \|\omega\|_{\mathcal{L}_2^q}^2 + \nu \|\tilde{x}_0\|^2}, \quad (3.17)$$

where  $\mu > 0$ . The term  $\sqrt{\mu}$  indicates the disturbance attenuation level, and  $\nu > 0$  is to be estimated.

In order to investigate the  $\mathcal{H}_\infty$  stability of the error dynamic (3.15), a subsequent quadratic Lyapunov function is considered:

$$V(\tilde{x}) = \tilde{x}^\top P \tilde{x}, \text{ where } P > 0 \in \mathbf{S}^n. \quad (3.18)$$

The derivative of  $V(\tilde{x})$  is computed along the trajectories of (3.15), yielding the subsequent outcome:

$$\begin{aligned} \dot{V}(\tilde{x}) = & \tilde{x}^\top (\mathcal{A}^\top P + P \mathcal{A}) \tilde{x} + \tilde{x}^\top \left[ \left( \sum_{i,j=1}^{m,\bar{n}} f_{ij} P G \mathcal{H}_{ij} H_i \right) + \left( \sum_{i,j=1}^{m,\bar{n}} f_{ij} P G \mathcal{H}_{ij} H_i \right)^\top \right] \tilde{x} \\ & - \tilde{x}^\top \left[ \left( \sum_{i,j=1}^{r,\bar{p}} g_{ij} P L F \mathcal{G}_{ij} G_i \right) + \left( \sum_{i,j=1}^{r,\bar{p}} g_{ij} P L F \mathcal{G}_{ij} G_i \right)^\top \right] \tilde{x} + \tilde{x}^\top (P \mathbb{E}) \omega + \omega^\top (\mathbb{E}^\top P) \tilde{x}. \end{aligned} \quad (3.19)$$

The error dynamic (3.15) fulfil the  $\mathcal{H}_\infty$  criterion (3.17) if it admits a Lyapunov function (3.18) such that

$$\mathcal{W} \triangleq \dot{V}(\tilde{x}) + \|\tilde{x}\|^2 - \mu \|\omega\|^2 \leq 0. \quad (3.20)$$

One can go through Remark 3.3 for more details about this.

**Remark 3.3**

If the inequality (3.20) is true, then it is easy to obtain:

$$\int_0^t \left( \dot{V}(\tilde{x}, \tau) + \|\tilde{x}(\tau)\|^2 - \mu \|\omega(\tau)\|^2 \right) d\tau \leq 0 \quad (3.21)$$

Since  $V(\tilde{x}(t)) \geq 0$ ,  $\forall t \geq 0$ , one can derive the following inequality from (3.21):

$$-V(\tilde{x}_0) + \|\tilde{x}\|_{\mathcal{L}_2^n}^2 - \mu \|\omega\|_{\mathcal{L}_2^q}^2 \leq 0 \text{ when } t \rightarrow \infty. \quad (3.22)$$

Further, the inequality (3.22) can be rewritten as:

$$\|\tilde{x}\|_{\mathcal{L}_2^n}^2 \leq \mu \|\omega\|_{\mathcal{L}_2^q}^2 + V(\tilde{x}_0). \quad (3.23)$$

Since  $V(\tilde{x}_0) \leq \lambda_{\max}(P) \|\tilde{x}_0\|^2$ , the condition specified in (3.23) is reformulated into:

$$\|\tilde{x}\|_{\mathcal{L}_2^n}^2 \leq \mu \|\omega\|_{\mathcal{L}_2^q}^2 + \lambda_{\max}(P) \|\tilde{x}_0\|^2. \quad (3.24)$$

The inequality (3.24) is identical to (3.17) along with  $\nu = \lambda_{\max}(P) > 0$ . Hence, the error dynamic (3.15) satisfies the  $\mathcal{H}_\infty$  criterion (3.17) if it possess a Lyapunov function (3.18) which holds the condition (3.20).

Now, through the utilisation of (3.20) and (3.19), we get:

$$\begin{aligned} \mathcal{W} &= \begin{bmatrix} \tilde{x} \\ \omega \end{bmatrix}^\top \begin{bmatrix} A^\top P + PA - PLC - C^\top L^\top P + \mathbb{I}_n & PE - PLD \\ (\star) & -\mu \mathbb{I}_q \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \omega \end{bmatrix} \\ &+ \begin{bmatrix} \tilde{x} \\ \omega \end{bmatrix}^\top \left[ \sum_{i,j=1}^{m,\bar{n}} \left( \begin{bmatrix} (PG\mathcal{H}_{ij}) \\ \mathbb{O} \end{bmatrix} \begin{bmatrix} f_{ij} H_i & \mathbb{O} \end{bmatrix} + \begin{bmatrix} f_{ij} H_i & \mathbb{O} \end{bmatrix}^\top \begin{bmatrix} (PG\mathcal{H}_{ij}) \\ \mathbb{O} \end{bmatrix}^\top \right) \right] \begin{bmatrix} \tilde{x} \\ \omega \end{bmatrix} \\ &+ \begin{bmatrix} \tilde{x} \\ \omega \end{bmatrix}^\top \left[ \sum_{i,j=1}^{r,\bar{p}} \left( \begin{bmatrix} (-PLF\mathcal{G}_{ij}) \\ \mathbb{O} \end{bmatrix} \begin{bmatrix} g_{ij} G_i & \mathbb{O} \end{bmatrix} + \begin{bmatrix} g_{ij} G_i & \mathbb{O} \end{bmatrix}^\top \begin{bmatrix} (-PLF\mathcal{G}_{ij}) \\ \mathbb{O} \end{bmatrix}^\top \right) \right] \begin{bmatrix} \tilde{x} \\ \omega \end{bmatrix}. \end{aligned} \quad (3.25)$$

Thus,  $\mathcal{W} \leq 0$  if

$$\mathcal{A}_L + \sum_{i,j=1}^{m,\bar{n}} \underbrace{\begin{bmatrix} (PG\mathcal{H}_{ij}) \\ \mathbb{O} \end{bmatrix}}_{U_{ij}^\top} \underbrace{\begin{bmatrix} H_i & \mathbb{O} \end{bmatrix}}_{V_{ij}} + V_{ij}^\top U_{ij} + \sum_{i,j=1}^{r,\bar{p}} \underbrace{\begin{bmatrix} (-R^\top F\mathcal{G}_{ij}) \\ \mathbb{O} \end{bmatrix}}_{M_{ij}^\top} \underbrace{\begin{bmatrix} G_i & \mathbb{O} \end{bmatrix}}_{N_{ij}} + N_{ij}^\top M_{ij} \leq 0, \quad (3.26)$$

where

$$\mathcal{A}_L = \begin{bmatrix} A^\top P + PA - R^\top C - C^\top R + \mathbb{I}_n & PE - R^\top D \\ (\star) & -\mu \mathbb{I}_q \end{bmatrix}, \quad (3.27)$$

and  $R^\top = PL$ .

Various LMI-based approaches have been developed in the literature [12–15, 21]. Each method is formulated by using different mathematical tools and provides improved LMI conditions.

Despite advances in this area of LMI relaxations, the resulting LMIs remain conservative. It indicates that there is room for more enhancements.

To enhance comprehensibility and reduce complexity, the ensuing notations are introduced:

$$U = \left[ U_{11}^\top \quad \cdots \quad U_{1\bar{n}}^\top \quad \cdots \quad U_{m1}^\top \quad \cdots \quad U_{m\bar{n}}^\top \right]^\top, \quad (3.28)$$

$$V = \left[ V_{11}^\top \quad \cdots \quad V_{1\bar{n}}^\top \quad \cdots \quad V_{m1}^\top \quad \cdots \quad V_{m\bar{n}}^\top \right]^\top, \quad (3.29)$$

$$M = \left[ M_{11}^\top \quad \cdots \quad M_{1\bar{p}}^\top \quad \cdots \quad M_{r1}^\top \quad \cdots \quad M_{r\bar{p}}^\top \right]^\top, \quad (3.30)$$

$$N = \left[ N_{11}^\top \quad \cdots \quad N_{1\bar{p}}^\top \quad \cdots \quad N_{r1}^\top \quad \cdots \quad N_{r\bar{p}}^\top \right]^\top, \quad (3.31)$$

where  $U_{ij}$ ,  $V_{ij}$ ,  $M_{ij}$  and  $N_{ij}$  are defined in (3.26). Further,  $V$  and  $N$  are expressed as follows:

$$V = \begin{bmatrix} V_{11} \\ \vdots \\ V_{1\bar{n}} \\ \vdots \\ V_{m1} \\ \vdots \\ V_{m\bar{n}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbb{H}_1 & 0 & \cdots & \mathbb{O} & \cdots & \mathbb{O} & \cdots & \mathbb{O} \\ \star & \mathbb{H}_1 & \cdots & \mathbb{O} & \cdots & \mathbb{O} & \cdots & \mathbb{O} \\ \star & \star & \ddots & \mathbb{O} & \cdots & \mathbb{O} & \cdots & \mathbb{O} \\ \star & \star & \star & \mathbb{H}_1 & \cdots & \mathbb{O} & \cdots & \mathbb{O} \\ \star & \star & \star & \star & \ddots & \mathbb{O} & \cdots & \mathbb{O} \\ \star & \star & \star & \star & \star & \mathbb{H}_m & \cdots & \mathbb{O} \\ \star & \star & \star & \star & \star & \star & \ddots & \mathbb{O} \\ \star & \mathbb{H}_m \end{bmatrix}}_{\mathbb{H}} \underbrace{\begin{bmatrix} f_{11}\mathbb{I} \\ f_{12}\mathbb{I} \\ \vdots \\ f_{1\bar{n}}\mathbb{I} \\ \vdots \\ f_{m1}\mathbb{I} \\ \vdots \\ f_{m\bar{n}}\mathbb{I} \end{bmatrix}}_{\Phi} = \mathbb{H}\Phi \quad (3.32)$$

and

$$N = \begin{bmatrix} N_{11} \\ \vdots \\ N_{1\bar{p}} \\ \vdots \\ N_{r1} \\ \vdots \\ N_{r\bar{p}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbb{G}_1 & \mathbb{O} & \cdots & \mathbb{O} & \cdots & \mathbb{O} & \cdots & \mathbb{O} \\ \star & \mathbb{G}_1 & \cdots & \mathbb{O} & \cdots & \mathbb{O} & \cdots & \mathbb{O} \\ \star & \star & \ddots & \mathbb{O} & \cdots & \mathbb{O} & \cdots & \mathbb{O} \\ \star & \star & \star & \mathbb{G}_1 & \cdots & \mathbb{O} & \cdots & \mathbb{O} \\ \star & \star & \star & \star & \ddots & \mathbb{O} & \cdots & \mathbb{O} \\ \star & \star & \star & \star & \star & \mathbb{G}_r & \cdots & \mathbb{O} \\ \star & \star & \star & \star & \star & \star & \ddots & \mathbb{O} \\ \star & \mathbb{G}_r \end{bmatrix}}_{\mathbb{G}} \underbrace{\begin{bmatrix} g_{11}\mathbb{I} \\ g_{12}\mathbb{I} \\ \vdots \\ g_{1\bar{p}}\mathbb{I} \\ \vdots \\ g_{r1}\mathbb{I} \\ \vdots \\ g_{r\bar{p}}\mathbb{I} \end{bmatrix}}_{\Psi} = \mathbb{G}\Psi \quad (3.33)$$

By deploying these aforementioned notations, the inequality (3.26) is reformulate as:

$$\mathcal{A}_L + \underbrace{U^\top (\mathbb{H}\Phi) + (\mathbb{H}\Phi)^\top U}_{\mathcal{N}_1} + \underbrace{M^\top (\mathbb{G}\Psi) + (\mathbb{G}\Psi)^\top M}_{\mathcal{N}_2} \leq 0. \quad (3.34)$$

In the sequel, two novel LMI techniques will be proposed by exploiting Young inequalities (A.9) and (A.10) as well as Lemma 2.2.

### 3.2.2 LMI formulation with Young standard inequality

This subsection is dedicated to the development of the LMI condition by utilising the Young inequality (A.9). It is often used in the literature for observer design tasks. In [20], this inequality

is deployed with  $Z = \delta \mathbb{I}$ ,  $\delta > 0$ , while the authors of [13] used a diagonal form of matrix  $Z$ . In this segment, the authors will use (A.9) with newly defined matrix multipliers (i.e., (2.82) and (2.83)). The implementation of the Young inequality (A.9) on the terms  $\mathcal{N}_1$  and  $\mathcal{N}_2$  yields:

$$\mathcal{N}_1 \leq U^\top Z^{-1} U + \Phi^\top \mathbb{H}^\top Z \mathbb{H} \Phi, \quad (3.35)$$

$$\mathcal{N}_2 \leq M^\top \mathbb{S}^{-1} M + \Psi^\top \mathbb{G}^\top \mathbb{S} \mathbb{G} \Psi, \quad (3.36)$$

where the matrices  $Z$  and  $\mathbb{S}$  are described in (2.82) and (2.83), respectively.

Consider  $\Phi_m$  and  $\Psi_m$  as described below:

$$\Phi_m^\top = \begin{bmatrix} f_{b_{11}} \mathbb{I} & \dots & f_{b_{1\bar{n}}} \mathbb{I} & \dots & f_{b_{m1}} \mathbb{I} & \dots & f_{b_{m\bar{n}}} \mathbb{I} \end{bmatrix}, \quad (3.37)$$

$$\Psi_m^\top = \begin{bmatrix} g_{b_{11}} \mathbb{I} & \dots & g_{b_{1p_1}} \mathbb{I} & \dots & g_{b_{r1}} \mathbb{I} & \dots & g_{b_{r\bar{p}}} \mathbb{I} \end{bmatrix}. \quad (3.38)$$

Since  $Z > 0$  and  $\mathbb{S} > 0$ , we can deduce the subsequent inequalities:

$$\mathbb{H}^\top Z \mathbb{H} \geq 0 \text{ and } \mathbb{G}^\top \mathbb{S} \mathbb{G} \geq 0.$$

As each element inside  $\Phi$  and  $\Phi_m$  admits the condition (3.13), Corollary 2.2 is deployed on  $\Phi$  and  $\Phi_m$ , and the ensuing inequality is derived:

$$\Phi^\top \mathbb{H}^\top Z \mathbb{H} \Phi \leq \Phi_m^\top \mathbb{H}^\top Z \mathbb{H} \Phi_m. \quad (3.39)$$

Analogously, for  $\Psi$  and  $\Psi_m$ , one can achieve:

$$\Psi^\top \mathbb{G}^\top \mathbb{S} \mathbb{H} \Phi \leq \Psi_m^\top \mathbb{G}^\top \mathbb{S} \mathbb{G} \Psi_m. \quad (3.40)$$

By utilising (3.39) and (3.40), inequalities (3.35) and (3.36) are rewritten as:

$$\mathcal{N}_1 \leq U^\top Z^{-1} U + \Phi_m^\top \mathbb{H}^\top Z \mathbb{H} \Phi_m, \quad (3.41)$$

$$\mathcal{N}_2 \leq M^\top \mathbb{S}^{-1} M + \Psi_m^\top \mathbb{G}^\top \mathbb{S} \mathbb{G} \Psi_m. \quad (3.42)$$

Hence, the condition specified in (3.34) is altered as:

$$\mathcal{A}_L + U^\top Z^{-1} U + \Phi_m^\top \mathbb{H}^\top Z \mathbb{H} \Phi_m + M^\top \mathbb{S}^{-1} M + \Psi_m^\top \mathbb{G}^\top \mathbb{S} \mathbb{G} \Psi_m \leq 0. \quad (3.43)$$

### Theorem 3.1

If there exist the matrices  $Z$ ,  $\mathbb{S}$ , expressed as in (2.82) and (2.83), respectively, along with the matrices  $P \in \mathbf{S}^n$  and  $R \in \mathbb{R}^{p \times n}$  and positive scalar  $\mu$ , such that the subsequent optimization problem is solvable:

$$\begin{aligned} & \min \mu \text{ subject to} \\ & \begin{bmatrix} \mathcal{A}_L & U^\top & \Phi_m^\top \mathbb{H}^\top Z & M^\top & \Psi_m^\top \mathbb{G}^\top \mathbb{S} \\ (\star) & -Z & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ (\star) & (\star) & -Z & \mathbb{O} & \mathbb{O} \\ (\star) & (\star) & (\star) & -\mathbb{S} & \mathbb{O} \\ (\star) & (\star) & (\star) & (\star) & -\mathbb{S} \end{bmatrix} < 0, \end{aligned} \quad (3.44)$$

where the matrices  $\mathcal{A}_L, U, M, \mathbb{H}, \mathbb{G}, \Phi_m$  and  $\Psi_m$  are depicted in (3.27), (3.28), (3.30), (3.32), (3.33), (3.37) and (3.38), respectively. Then, the error dynamic (3.15) fulfils  $\mathcal{H}_\infty$

criterion (3.17) with  $\nu = \lambda_{\max}(P)$  and minimum  $\mu$  obtained from the solution of LMI (3.44). The observer gain is computed as  $L = P^{-1}R^\top$ .

**Proof:** One can easily derive LMI (3.44) by implementing Schur Lemma A.3 on (3.43). Hence, it ensure that the error dynamics (3.15) satisfies  $\mathcal{H}_\infty$  criterion (3.17). This ends the proof.  $\square$

In the subsequent segment of this section, the establishment of the enhanced LMI condition is illustrated by exploring a new variant of the Young inequality (A.10) and the well-known LPV approach.

### 3.2.3 An Enhanced LMI criterion: exploiting LPV approach

Recently, the authors of [14, 22] incorporated a new variant of Young inequality (A.10) into the development of LMI-based observers, yielding the enhanced LMI conditions. In this part of the chapter, the formulation of a new LPV-based LMI is elaborated.

The deployment of the new variant of Young inequality (A.10) on the terms  $\mathcal{N}_1$  and  $\mathcal{N}_2$  of (3.34) resulted in the ensuing inequalities:

$$\mathcal{N}_1 \leq \frac{1}{2}(U + \mathbb{Z}\mathbb{H}\Phi)^\top \mathbb{Z}^{-1}(U + \mathbb{Z}\mathbb{H}\Phi), \quad (3.45)$$

$$\mathcal{N}_2 \leq \frac{1}{2}(M + \mathbb{S}\mathbb{G}\Psi)^\top \mathbb{S}^{-1}(M + \mathbb{S}\mathbb{G}\Psi), \quad (3.46)$$

where the matrices  $\mathbb{Z}$  and  $\mathbb{S}$  are specified in (2.82) and (2.83), respectively. The substitution of (3.45) and (3.46) into the inequality (3.34) yields:

$$\mathcal{A}_L + \frac{1}{2}(U + \mathbb{Z}\mathbb{H}\Phi)^\top \mathbb{Z}^{-1}(U + \mathbb{Z}\mathbb{H}\Phi) + \frac{1}{2}(M + \mathbb{S}\mathbb{G}\Psi)^\top \mathbb{S}^{-1}(M + \mathbb{S}\mathbb{G}\Psi) \leq 0. \quad (3.47)$$

Each element inside  $\Phi$  and  $\Psi$  fulfil (3.13) and (3.14). Therefore, every single component of  $\Phi$  and  $\Psi$  admits membership in convex sets  $\mathcal{V}_\phi$  and  $\mathcal{V}_\psi$ , respectively. These two sets,  $\mathcal{V}_\phi$  and  $\mathcal{V}_\psi$ , are specified as follows:

$$\mathcal{V}_\phi \triangleq \{\Phi : 0 \leq f_{ij} \leq f_{b_{ij}}, \forall i \in \{1, \dots, m\} \& j \in \{1, \dots, \bar{n}\}\}, \quad (3.48)$$

$$\mathcal{V}_\psi \triangleq \{\Psi : 0 \leq g_{ij} \leq g_{b_{ij}}, \forall i \in \{1, \dots, r\} \& j \in \{1, \dots, \bar{p}\}\}. \quad (3.49)$$

Further, the sets of vertices of each set  $\mathcal{V}_\phi$  and  $\mathcal{V}_\psi$  are described as:

$$\mathcal{H}_\phi = \left\{ \{\mathcal{H}_\phi^{11}, \dots, \mathcal{H}_\phi^{1\bar{n}}, \dots, \mathcal{H}_\phi^{m1}, \dots, \mathcal{H}_\phi^{m\bar{n}}\} : \mathcal{H}_\phi^{ij} \in [0, f_{b_{ij}}] \right\}, \quad (3.50)$$

$$\mathcal{H}_\psi = \left\{ \{\mathcal{H}_\psi^{11}, \dots, \mathcal{H}_\psi^{1\bar{p}}, \dots, \mathcal{H}_\psi^{r1}, \dots, \mathcal{H}_\psi^{r\bar{p}}\} : \mathcal{H}_\psi^{ij} \in [0, g_{b_{ij}}] \right\}. \quad (3.51)$$

Therefore, the inequality (3.47) is modified as:

$$\begin{aligned} & \mathcal{A}_L + \left[ \frac{1}{2}(U + \mathbb{Z}\mathbb{H}\Phi)^\top \mathbb{Z}^{-1}(U + \mathbb{Z}\mathbb{H}\Phi) \right]_{\forall \Phi \in \mathcal{H}_\phi} \\ & + \left[ \frac{1}{2}(M + \mathbb{S}\mathbb{G}\Psi)^\top \mathbb{S}^{-1}(M + \mathbb{S}\mathbb{G}\Psi) \right]_{\forall \Psi \in \mathcal{H}_\psi} \leq 0. \end{aligned} \quad (3.52)$$

Now, we can state the following theorem.

**Theorem 3.2**

Let us consider the matrices  $P > 0 \in \mathbf{S}^n$ ,  $R \in \mathbb{R}^{p \times n}$ ,  $Z$  in the form of (2.82),  $S$  as illustrated in (2.83), and the ensuing subsequent optimization problem:

$$\begin{aligned} & \min \mu \text{ subject to} \\ & \begin{bmatrix} \mathcal{A}_L & (U + ZH\Phi)^\top & (M + SG\Psi)^\top \\ (\star) & -2Z & \mathbb{O} \\ (\star) & (\star) & -2S \end{bmatrix} < 0, \quad \forall \Phi \in \mathcal{H}_\phi, \quad \forall \Psi \in \mathcal{H}_\psi, \end{aligned} \quad (3.53)$$

where all variables and parameters remain consistent with those described in Theorem 3.1. If the aforementioned LMI (3.53) is feasible, then the error dynamic (3.15) is  $\mathcal{H}_\infty$  asymptotically stable satisfying (3.17) with  $\nu = \lambda_{\max}(P)$  and minimum  $\mu$  obtained from the solution of LMI (3.44). The gain matrix  $L$  is determined by using  $L = P^{-1}R^\top$ .

**Proof:** The outcome of the implementation of the Schur Lemma A.3 on the expression (3.52) is represented by the LMI (3.53). Further, from convexity principal [18], if (3.53) is resolved for all  $\Phi \in \mathcal{H}_\phi$  and  $\Psi \in \mathcal{H}_\psi$ , then the error dynamic (3.15) holds  $\mathcal{H}_\infty$  criterion (3.17) which ensures the asymptotic convergence of the error dynamic.  $\square$

Both LMIs (3.44) and (3.53) are able to assure the exponential convergence of estimation error dynamic (3.15) in the absence of the external disturbances/noises, i.e.,  $\omega = 0$ , which is demonstrated in the subsequent remark:

**Remark 3.4**

At  $\omega = 0$ , the inequality (3.20) becomes:

$$\dot{V}(\tilde{x}) + \|\tilde{x}\|^2 \leq 0, \quad (3.54)$$

which is equivalent to the exponential stability condition (2.45) along with  $\epsilon = 1$ . From (3.18),  $V(\tilde{x})$  admits (2.44). Therefore,  $V(\tilde{x})$  satisfy (1.26) and (1.27). Since the error dynamic (3.15) admits  $V(\tilde{x})$  that fulfils all the conditions specified in Theorem 1.2, it is exponentially stable when  $\omega = 0$ .

In the subsequent part of this section, the development of the observer for the nonlinear systems having linear outputs is discussed.

### 3.2.4 $\mathcal{H}_\infty$ observers for the nonlinear systems with linear outputs

The ensuing part is dedicated to the design of observers for a class of disturbance-affected nonlinear systems with linear outputs. The explicit structure of these systems is illustrated as:

$$\begin{aligned} \dot{x} &= Ax + Gf(x) + Bu + E\omega, \\ y &= Cx + D\omega, \end{aligned} \quad (3.55)$$

where all variables and parameters are the same as the one depicted in (3.2). The nonlinear part of the dynamics of (3.55), i.e.,  $f(x)$ , is assumed to be globally Lipschitz and it admits the detailed form (2.34).

For the purpose of the state estimation of the system (3.55), the following observer structure is employed:

$$\dot{\hat{x}} = A\hat{x} + G \begin{bmatrix} f_1(H_1\hat{x}) \\ f_2(H_2\hat{x}) \\ \vdots \\ \underbrace{f_i(H_i\hat{x})}_{\hat{v}_i} \\ \vdots \\ f_m(H_m\hat{x}) \end{bmatrix} + Bu + L(y - C\hat{x}), \quad (3.56)$$

where  $\hat{x}$  denotes the estimate of state  $x$ , and the matrix  $L \in \mathbb{R}^{n \times p}$  represents the observer gain. Analogous to Section (3.2.1), the estimation error of the observer (3.56) is defined as:  $\tilde{x} = x - \hat{x}$ . Further, through the utilisation of (3.55) and (3.56), the error dynamic is computed, and illustrated as:

$$\dot{\tilde{x}} = \mathcal{A}\tilde{x} + G\tilde{f}(x, \hat{x}) + \mathbb{E}\omega, \quad (3.57)$$

where  $\mathcal{A}$ ,  $\mathbb{E}$  and  $\tilde{f}(x, \hat{x})$  are specified in (3.5), (3.6) and (3.7), respectively. Similar to the aforementioned section, the objective of the LMI formulation is to determine the observer gain matrix  $L$  such that the estimation error dynamic (3.57) fulfils  $\mathcal{H}_\infty$  criterion (3.17). The subsequent corollaries present LMI conditions that ensure the  $\mathcal{H}_\infty$  asymptotic stability of the estimation error dynamic (3.57).

### Corollary 3.1

If there exist the matrices  $\mathbb{Z}$  specified in (2.82),  $P > 0 \in \mathbf{S}^n$ ,  $R \in \mathbb{R}^{p \times n}$ , and a positive scalar  $\mu$ , such that the following optimisation problem is solvable:

$$\begin{aligned} & \min \mu \text{ subject to} \\ & \begin{bmatrix} \mathcal{A}_L & U^\top & \Phi_m^\top \mathbb{H}^\top \mathbb{Z} \\ (\star) & -\mathbb{Z} & \mathbb{O} \\ (\star) & (\star) & -\mathbb{Z} \end{bmatrix} < 0, \end{aligned} \quad (3.58)$$

where the matrices  $\mathcal{A}_L$ ,  $U$ ,  $\mathbb{H}$ , and  $\Phi_m$  are depicted in (3.27), (3.28), (3.32), and (3.37), respectively. Then,  $\mathcal{H}_\infty$  criterion (3.17) is fulfilled by the error dynamic (3.57). The observer gain is given by  $L = P^{-1}R^\top$ .

**Proof:** One can follow the steps provided in the proof of Theorem 3.1 to derive LMI (3.58).  $\square$

Further, the reformulation of the LPV-based LMI (3.53) to ensure the stabilisation of (3.57) is stated in the following corollary.

### Corollary 3.2

Let us consider the matrix  $\mathbb{Z}$  in the form shown in (2.82). The error dynamic (3.57) satisfies  $\mathcal{H}_\infty$  criterion (3.17) if there exist the matrices  $P > 0 \in \mathbf{S}^n$ ,  $R \in \mathbb{R}^{p \times n}$  and positive scalar  $\mu$

such that the subsequent optimisation problem is solvable:

$$\begin{aligned} & \min \mu \text{ subject to} \\ & \begin{bmatrix} \mathcal{A}_L & (U + \mathbb{Z}\mathbb{H}\Phi)^\top \\ (\star) & -2\mathbb{Z} \end{bmatrix} < 0, \forall \Phi \in \mathcal{H}_\phi, \end{aligned} \quad (3.59)$$

where the matrices  $\mathcal{A}_L, U, \mathbb{H}$ , and  $\mathbb{Z}$  are described in (3.27), (3.28), (3.35) and (2.82), respectively.

**Proof:** One can emulate the proof of Theorem 3.2 in order to prove the aforementioned Corollary.  $\square$

In the sequel, the development of a nonlinear observer for non-global Lipschitz systems using the proposed matrix-multiplier-based LMI approach is investigated.

### 3.3 LMI-based observers for locally Lipschitz systems: exploiting an ISS framework

Although all the proposed observer methodologies mentioned in Section 3.2 provide efficient state estimations, it has several obstacles such as assumptions about the systems. One of the explicit assumptions for the nonlinear observer design is the existence of global Lipschitz nonlinearities in the systems. All these proposed observers fail to perform the state estimation task in the local Lipschitz nonlinear systems. The primary agenda of this segment is to provide a novel observer methodology for a class of non-global Lipschitz nonlinear systems. In the previous segment of the chapter,  $\mathcal{H}_\infty$  criterion-based observer design is proposed for the estimation of the state in the presence of noise. An alternative for an  $\mathcal{H}_\infty$  criterion is the use of the ISS notion. The concept of ISS is recapitulated in Section 1.2.5. This section is focused on the development of a novel LMI condition to design the observers for a class of non-globally Lipschitz systems.

#### 3.3.1 Insight on the Hilbert projection

An overview of the Hilbert projection is provided in this part of the chapter. The authors of [88] had proposed the famous **Hilbert projection theorem**, which is illustrated as follows:

##### Theorem 3.3: Hilbert projection theorem [88]

Consider a Hilbert space  $\mathcal{H}$  and nonempty closed convex sets  $\Omega \subset \mathcal{H}$ . For any vector  $a$  in  $\mathcal{H}$ , there exists a unique vector  $c \in \Omega$  which fulfils the subsequent inequality:

$$\|a - c\| = \min_{b \in \Omega} \|a - b\|. \quad (3.60)$$

Then, such vector  $c$  is known as the **projection of  $a$  on  $\Omega$** .

Theorem 3.3 is a cornerstone element in functional analysis. The essential parameter for the existence and uniqueness of the Hilbert projection is the closeness and convexity of set  $\Omega$ . One can consider Hilbert space as  $\mathcal{H} = \mathbb{R}^n$  or  $\mathcal{H} = \mathbb{R}$  in case of scalar variables. Through the utilisation of this theorem, the ensuing definition is established:

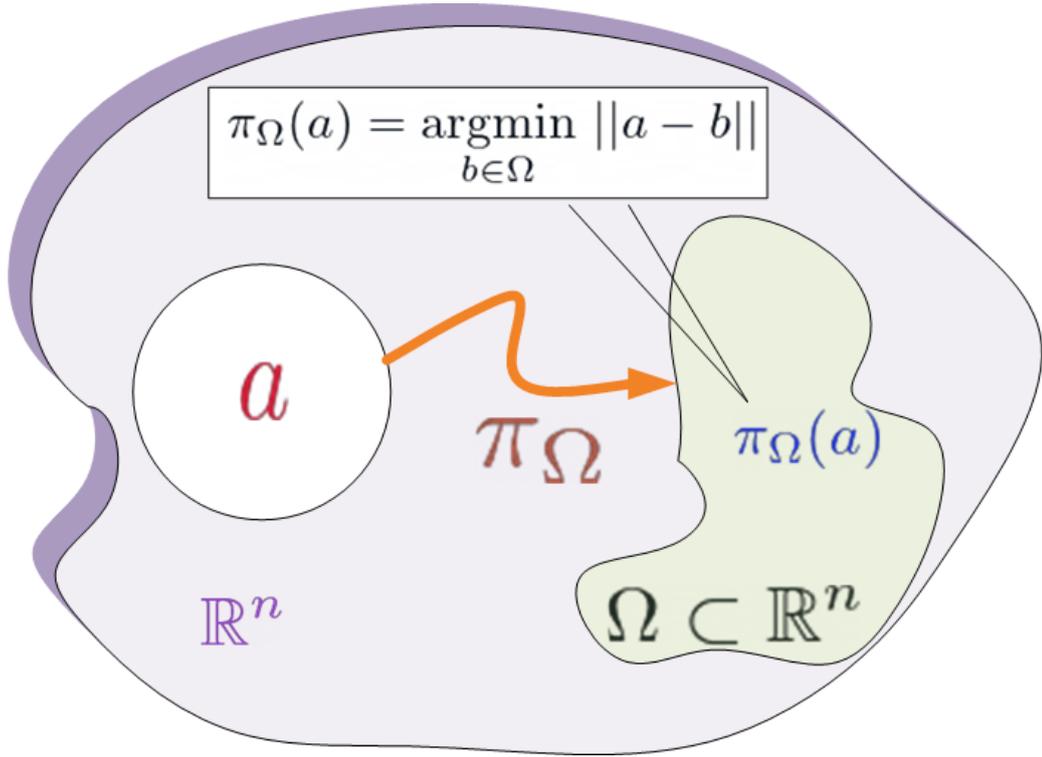


Figure 3.2: Depiction of Hilbert projection

**Definition 3.1: Projection  $\pi_\Omega(a)$** 

For a closed convex and nonempty set  $\Omega \subset \mathbb{R}^n$ , there exists a linear application  $\pi_\Omega : \mathbb{R}^n \rightarrow \Omega$ , which is called as **a projection on the convex set  $\Omega$** , and it holds:

$$\pi_\Omega(a) = \operatorname{argmin}_{b \in \Omega} \|a - b\|, \quad (3.61)$$

where  $a, b, c \in \mathbb{R}^n$  are vectors.

Throughout this section, the notation  $\pi_\Omega(a)$  is used to represent a projection of the vector  $a \in \mathbb{R}^n$  on a closed convex set  $\Omega$ . The schematic representation of the Hilbert projection is portrayed in Figure 3.2. The term  $\pi_\Omega(a)$  is a distinct vector in  $\Omega$ , and it fulfils the succeeding inequalities:

$$\|a - \pi_\Omega(a)\| \leq \|a - b\|, \forall b \in \Omega \quad (3.62)$$

and

$$\langle a - \pi_\Omega(a), b - \pi_\Omega(a) \rangle \leq 0, \forall b \in \Omega. \quad (3.63)$$

The aforementioned conditions are utilised to prove some useful properties of  $\pi_\Omega(\cdot)$ . The vector  $\pi_\Omega(\cdot)$  encompasses certain characteristics that play a vital role in the observer design. These advantageous attributes are listed as follows:

1. The projection  $\pi_\Omega$  is **idempotent**, that is,

$$\pi_\Omega \circ \pi_\Omega = \pi_\Omega.$$

This property is easy to prove by using (3.62).  $\forall a \in \Omega$  and  $a = b$ , the inequality (3.62) is reformulated as:  $\|a - \pi_\Omega(a)\| = a - a = 0$ . It leads to  $\pi_\Omega(a) = a$ . Additionally, we obtain:  $\pi_\Omega \circ \pi_\Omega = \pi_\Omega(\pi_\Omega(a)) = \pi_\Omega(a)$ .

2. The linear application  $\pi_\Omega$  is **monotonic** within the subsequent context:

$$\frac{\pi_\Omega(a) - \pi_\Omega(b)}{a - b} \geq 0, \forall a, b \in \mathbb{R} \text{ and } a \neq b. \quad (3.64)$$

One can prove this condition by utilising (3.63) and (3.61).

3. The projection  $\pi_\Omega$  is a **Lipschitz function** with **1 as the Lipschitz constant**. In simple words, it is a **1-Lipschitz function**, i.e.,

$$\|\pi_\Omega(a) - \pi_\Omega(b)\| \leq \|a - b\|, \forall a, b \in \mathbb{R}^n. \quad (3.65)$$

With reference to the above specified features of the  $\pi_\Omega$ , Lemma 2.1 is reformulated by the authors of [23], which is as follows:

**Lemma 3.1:** [23]

Let us consider a nonlinear function  $\phi: \mathbb{R}^n \rightarrow \mathbb{R}^n$ , and it is Lipschitz on a convex closed and non-empty set  $\Omega$ . Then, there exist functions  $\phi_{ij}^\Omega: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ , and constants  $\phi_{ij_{\min}}^\Omega$  and  $\phi_{ij_{\max}}^\Omega$  such that  $\forall a, b \in \mathbb{R}^n, a \neq b$ ,

$$\phi(a) - \phi(b) = \sum_{i=1}^n \sum_{j=1}^n \phi_{ij}^\Omega \mathcal{H}_{ij}(a - b), \quad (3.66)$$

where  $\mathcal{H}_{ij} = e_n(i)e_n^\top(j)$ , and  $\phi_{ij}^\Omega \triangleq \phi_{ij}^\Omega(a^{b_{j-1}}, a^{b_j})$ . The functions  $\phi_{ij}^\Omega(\cdot)$  are globally bounded as follows:

$$\phi_{ij_{\min}}^\Omega \leq \phi_{ij}^\Omega \leq \phi_{ij_{\max}}^\Omega. \quad (3.67)$$

In the sequel, Lemma 3.1 will be used in the LMI-based observer formulation.

The concept of ISS is recapitulated in Section 1.2.5. The necessary criteria are illustrated in Definitions 19 and 20 of Chapter 1. In the subsequent section, these ISS conditions are employed in the LMI formulation in order to obtain a sufficient condition for the observer design.

### 3.3.2 ISS-based observer design: articulating the problem statement

This section is dedicated to the intricate establishment of the observer which is specifically tailored for the disturbance-affected locally Lipschitz nonlinear systems.

For the simplicity and lucidity of presentation, first, a class of nonlinear systems having nonlinear outputs is deployed for observer design purposes.

Let us consider the ensuing form of the nonlinear system:

$$\begin{aligned} \dot{x} &= Ax + Gf(x) + Bu + E\omega, \\ y &= Cx + Hh(x) + D\omega, \end{aligned} \quad (3.68)$$

where  $x \in \mathbb{R}^n$  and  $y \in \mathbb{R}^p$  are the states and measurements of the system, respectively, and  $u \in \mathbb{R}^s$  is the system input. The process dynamics and the outputs of the system are corrupted

by a  $\mathcal{L}_2$  bounded disturbances/noise, and it is denoted by  $\omega$ . The matrices  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times s}$ ,  $G \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{p \times n}$ , and  $H \in \mathbb{R}^{p \times r}$ ,  $E \in \mathbb{R}^{n \times q}$  and  $D \in \mathbb{R}^{p \times q}$  are known and constant. Nonlinear functions  $f(\cdot)$  and  $h(\cdot)$  are expressed in the following detailed form:

$$f(x) = \begin{bmatrix} f_1(F_1x) \\ f_2(F_2x) \\ \vdots \\ f_m(F_mx) \end{bmatrix} \quad (3.69)$$

$$h(x) = \begin{bmatrix} h_1(H_1x) \\ h_2(H_2x) \\ \vdots \\ h_r(H_rx) \end{bmatrix}, \quad (3.70)$$

where  $F_i \in \mathbb{R}^{\bar{n} \times n}$ ,  $F_i x \in \Omega_{f_i} \forall i \in \{1, \dots, m\}$  and  $H_i \in \mathbb{R}^{\bar{p} \times n}$ ,  $H_i x \in \Omega_{h_i} \forall i \in \{1, \dots, r\}$ .

### Assumption 3.1

Let us presume that both sets  $\Omega_{f_i}$  and  $\Omega_{h_i}$  are positively invariant compact non-empty sets, i.e.,  $\Omega \neq \emptyset$ , and convex. Further, it is considered that  $f_i(\cdot)$  and  $h_i(\cdot)$  are locally Lipschitz-continuous functions in  $\Omega_{f_i}$  and  $\Omega_{h_i}$ , respectively.

For the state estimation of the systems (3.68), the observer form (3.3) is commonly used in the literature. This type of observer relies on the assumption that the nonlinearities present in the systems are globally Lipschitz. It is one of the major limitations of the standard Luenberger observers (3.3). Additionally, the criterion of the global Lipschitz property plays a crucial role in the Lyapunov analysis of the estimation error dynamics of the observer as well as in the LMI formulations by offering an upper bound to the term  $(f(x) - f(\hat{x}))$ . One can refer to [11, 32, 66] for more details about this. Since the nonlinear system (3.68) encompasses the non-global nonlinearities, the observer form specified in (3.3) is can not used for this model. Further, there are no such conditions that ensure that estimated states hold  $F_i \hat{x} \in \Omega_{f_i}$  and  $H_i \hat{x} \in \Omega_{h_i}$  even though system's state fulfil  $F_i x \in \Omega_{f_i}$  and  $H_i x \in \Omega_{h_i}$ . One of the solutions to mitigate the aforementioned drawback is proposed in [23]. The incorporation of the Hilbert projection on the non-global nonlinearities allows us to extend these nonlinearities as global Lipschitz nonlinearities in  $\mathbb{R}^n$ . The ensuing observer structure is proposed for the system (3.68):

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + G(f \circ \pi_{\Omega_f}(\hat{x})) + Bu + L(y - \hat{y}), \\ \hat{y} &= C\hat{x} + H(h \circ \pi_{\Omega_h}(\hat{x})), \end{aligned} \quad (3.71)$$

where  $f \circ \pi_{\Omega_f}(\hat{x})$  and  $h \circ \pi_{\Omega_h}(\hat{x})$  are defined as follows:

$$f \circ \pi_{\Omega_f}(\hat{x}) = \begin{bmatrix} f_1 \circ \pi_{\Omega_{f_1}}(F_1 \hat{x}) \\ f_2 \circ \pi_{\Omega_{f_2}}(F_2 \hat{x}) \\ \vdots \\ f_m \circ \pi_{\Omega_{f_m}}(F_m \hat{x}) \end{bmatrix}, \quad (3.72)$$

$$h \circ \pi_{\Omega_h}(\hat{x}) = \begin{bmatrix} h_1 \circ \pi_{\Omega_{h_1}}(H_1 \hat{x}) \\ h_2 \circ \pi_{\Omega_{h_2}}(H_2 \hat{x}) \\ \vdots \\ h_r \circ \pi_{\Omega_{h_r}}(H_r \hat{x}) \end{bmatrix}, \quad (3.73)$$

such that each  $f_i \circ \pi_{\Omega_{f_i}}(\cdot)$  and  $h_i \circ \pi_{\Omega_{h_i}}(\cdot)$  is globally Lipschitz in  $\mathbb{R}^n$ . The terms  $f_i \circ \pi_{\Omega_{f_i}}(\cdot)$  and  $h_i \circ \pi_{\Omega_{h_i}}(\cdot)$  are known as the Hilbert projection of  $H_i \hat{x}$  and  $F_i \hat{x}$  on the set  $\Omega_{f_i}$  and  $\Omega_{h_i}$ , respectively.

By using **idempotent property** of Hilbert projection, we obtain:

$$f_i(\pi_{\Omega_{f_i}}(F_i \hat{x})) = f_i(F_i \hat{x}) \quad \forall i \in \{1, \dots, m\} \quad \text{and} \quad h_i(\pi_{\Omega_{h_i}}(H_i \hat{x})) = h_i(H_i \hat{x}) \quad \forall i \in \{1, \dots, r\}.$$

It implies that applying Hilbert projection  $\pi_{\Omega}(\cdot)$  does not modify the system behaviour. For additional information about this, one can go through Definition 1 and [23].

The estimation error for the observer specified in (3.71) is defined as:

$$\tilde{x} = x - \hat{x}. \quad (3.74)$$

Through the utilisation of (3.68) and (3.71), the estimation error dynamic is obtained as follows:

$$\dot{\tilde{x}} = A_L \tilde{x} + G \Delta_f - LH \Delta_h + E_L \omega, \quad (3.75)$$

where

$$A_L = (A - LC), \quad (3.76)$$

$$E_L = (E - LD), \quad (3.77)$$

$\Delta_f = f(x) - f \circ \pi_{\Omega_f}(\hat{x})$  and  $\Delta_h = h(x) - h \circ \pi_{\Omega_h}(\hat{x})$ .

From (3.72) and (3.73),  $\Delta_f$  and  $\Delta_h$  are expressed in the subsequent form:

$$\Delta_f = f(x) - f \circ \pi_{\Omega_f}(\hat{x}) = \begin{bmatrix} f_1(F_1 x) \\ f_2(F_2 x) \\ \vdots \\ f_m(F_m x) \end{bmatrix} - \begin{bmatrix} f_1 \circ \pi_{\Omega_{f_1}}(F_1 \hat{x}) \\ f_2 \circ \pi_{\Omega_{f_2}}(F_2 \hat{x}) \\ \vdots \\ f_m \circ \pi_{\Omega_{f_m}}(F_m \hat{x}) \end{bmatrix}, \quad (3.78)$$

$$\Delta_h = h(x) - h \circ \pi_{\Omega_h}(\hat{x}) = \begin{bmatrix} h_1(H_1 x) \\ h_2(H_2 x) \\ \vdots \\ h_r(H_r x) \end{bmatrix} - \begin{bmatrix} h_1 \circ \pi_{\Omega_{h_1}}(H_1 \hat{x}) \\ h_2 \circ \pi_{\Omega_{h_2}}(H_2 \hat{x}) \\ \vdots \\ h_r \circ \pi_{\Omega_{h_r}}(H_r \hat{x}) \end{bmatrix}. \quad (3.79)$$

By deploying Lemma 3.1 on (3.78) and (3.79), we get:

$$\Delta_f = \sum_{i,j=1}^{m,\bar{n}} f_{ij}^{\Omega_{f_i}} \mathcal{F}_{ij} F_i \tilde{x}, \quad (3.80)$$

$$\Delta_h = \sum_{i,j=1}^{r,\bar{p}} h_{ij}^{\Omega_{h_i}} \mathcal{H}_{ij} H_i \tilde{x}, \quad (3.81)$$

where  $f_{ij}^{\Omega_{f_i}} : \mathbb{R}^{\bar{n}} \times \mathbb{R}^{\bar{n}} \rightarrow \mathbb{R}$ ,  $h_{ij}^{\Omega_{h_i}} : \mathbb{R}^{\bar{p}} \times \mathbb{R}^{\bar{p}} \rightarrow \mathbb{R}$  are functions such that  $f_{ij}^{\Omega_{f_i}} \triangleq f_{ij}^{\Omega}(X_i^{\hat{X}_{j-1}}, X_i^{\hat{X}_j})$ ,  $h_{ij}^{\Omega_{h_i}} \triangleq h_{ij}^{\Omega}(X_i^{\hat{X}_{j-1}}, X_i^{\hat{X}_j})$ . The functions  $f_{ij}^{\Omega_{f_i}}$ , and  $h_{ij}^{\Omega_{h_i}}$  hold  $f_{ij \min}^{\Omega} \leq f_{ij}^{\Omega} \leq f_{ij \max}^{\Omega}$  and  $h_{ij \min}^{\Omega} \leq h_{ij}^{\Omega} \leq h_{ij \max}^{\Omega}$ , respectively. Additionally,  $\mathcal{F}_{ij} = e_m(i) e_{\bar{n}}^{\top}(j)$  and  $\mathcal{H}_{ij} = e_r(i) e_{\bar{p}}^{\top}(j)$ .

Without loss of generality, let us assume that  $f_{ij \min}^{\Omega} = 0$  and  $h_{ij \min}^{\Omega} = 0$ , i.e.,

$$0 \leq f_{ij}^{\Omega} \leq f_{ij \max}^{\Omega}, \quad (3.82)$$

$$0 \leq h_{ij}^{\Omega} \leq h_{ij \max}^{\Omega}. \quad (3.83)$$

The additional information about this is elaborated in Remark 3.2 and [14].  
By using (3.80) and (3.81), the error dynamic (3.75) is reformulated as:

$$\dot{\tilde{x}} = A_L \tilde{x} + \sum_{i,j=1}^{m,\bar{n}} f_{ij}^\Omega G \mathcal{F}_{ij} F_i \tilde{x} - \sum_{i,j=1}^{r,\bar{p}} h_{ij}^\Omega L H \mathcal{H}_{ij} H_i \tilde{x} + E_L \omega. \quad (3.84)$$

Now, the key purpose of this section is outlined as follows:

#### Objective

The primary goal of this Section is to derive the necessary conditions such that the system (3.84) is ISS with respect to  $\omega$ .

### 3.3.3 LMI synthesis

In this part of the Section, the novel LMI condition is derived by incorporating the ISS notion with a popular LPV approach, which ensures the stability of the error dynamic (3.84) along with optimal noise attenuation.

For the clarity and lucidity of the presentation, this segment of the chapter is structured as follows:

1. First, the essential criteria to guarantee the ISS behaviour of the error dynamic (3.84) are derived by utilising the ISS Lyapunov functions.
2. Later on, the novel LMI condition is deduced in sub-unit 3.3.3.2 by deploying the aforementioned conditions.

#### 3.3.3.1 Formulating the essential conditions for ISS

The following theorem illustrates the required conditions that ensure the ISS behaviour of the system (3.84) w.r.t.  $\omega$ :

#### Theorem 3.4

- I) The error dynamic (3.84) is ISS with respect to  $\omega$  if it admits an ISS-Lyapunov function

$$V(\tilde{x}) = \tilde{x}^\top P \tilde{x}, \quad P = P^\top > 0 \in \mathbb{R}^{n \times n}. \quad (3.85)$$

- II) The trajectories of the system (3.84) fulfil:

$$\|\tilde{x}(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} e^{-\frac{\sigma}{2}t} \|\tilde{x}(0)\| + \sqrt{\frac{\delta}{\sigma \lambda_{\min}(P)}} \|\omega(\cdot)\|_\infty, \quad (3.86)$$

for any  $\mathcal{L}_\infty$  bounded noise  $\omega$ .

III) The estimation error  $\tilde{x}(t)$  is bounded when  $t$  approaches to  $\infty$ , that is,

$$\|\tilde{x}(\infty)\| \leq \sqrt{\frac{\delta}{\sigma \lambda_{\min}(P)}} \|\omega(\cdot)\|_{\infty}. \quad (3.87)$$

**Proof:** To enrich the comprehensibility, the proof of the theorem is divided into the following parts:

#### A. The proof for ISS-Lyapunov function:

Through the utilisation of (3.85),  $V(\tilde{x})$  admits:

$$\lambda_{\min}(P)\|\tilde{x}\|^2 \leq V(\tilde{x}) \leq \lambda_{\max}(P)\|\tilde{x}\|^2. \quad (3.88)$$

Further, the derivative of the Lyapunov function  $V(\tilde{x})$  is given by

$$\begin{aligned} \dot{V}(\tilde{x}) = & \tilde{x}^{\top} \left[ PA_L + A_L^{\top} P + \sum_{i,j=1}^{m,\bar{n}} \left( f_{ij}^{\Omega} PG\mathcal{F}_{ij}F_i + (f_{ij}^{\Omega} G\mathcal{F}_{ij}F_i)^{\top} P \right) \right. \\ & \left. - \sum_{i,j=1}^{r,\bar{p}} \left( h_{ij}^{\Omega} PLH\mathcal{H}_{ij}H_i + (h_{ij}^{\Omega} LH\mathcal{H}_{ij}H_i)^{\top} P \right) \right] \tilde{x} + \tilde{x}^{\top} PE_L\omega + \omega^{\top} E_L^{\top} P\tilde{x}. \end{aligned} \quad (3.89)$$

By implementing Young inequality A.9, we deduce:

$$\tilde{x}^{\top} PE_L\omega + \omega^{\top} E_L^{\top} P\tilde{x} \leq \delta \tilde{x}^{\top} (PE_L)^{\top} (PE_L) \tilde{x} + \delta \omega^{\top} \omega, \quad (3.90)$$

where  $\delta$  is a positive scalar. It leads to:

$$\begin{aligned} \dot{V}(\tilde{x}) \leq & \tilde{x}^{\top} \left[ PA_L + A_L^{\top} P + (PE_L)^{\top} (\delta \mathbb{I}) (PE_L) + \sum_{i,j=1}^{m,\bar{n}} \left( f_{ij}^{\Omega} PG\mathcal{F}_{ij}F_i \right. \right. \\ & \left. \left. + (f_{ij}^{\Omega} G\mathcal{F}_{ij}F_i)^{\top} P \right) - \sum_{i,j=1}^{r,\bar{p}} \left( h_{ij}^{\Omega} PLH\mathcal{H}_{ij}H_i + (h_{ij}^{\Omega} LH\mathcal{H}_{ij}H_i)^{\top} P \right) \right] \tilde{x} + \delta \omega^{\top} \omega. \end{aligned} \quad (3.91)$$

Further, inequality (3.91) is modified as:

$$\dot{V}(\tilde{x}) \leq \tilde{x}^{\top} Q \tilde{x} + \delta \omega^{\top} \omega, \quad (3.92)$$

where

$$\begin{aligned} Q = & PA_L + A_L^{\top} P + (PE_L)^{\top} (\delta \mathbb{I}) (PE_L) + \sum_{i,j=1}^{m,\bar{n}} \left( f_{ij}^{\Omega} PG\mathcal{F}_{ij}F_i + (f_{ij}^{\Omega} G\mathcal{F}_{ij}F_i)^{\top} P \right) \\ & - \sum_{i,j=1}^{r,\bar{p}} \left( h_{ij}^{\Omega} PLH\mathcal{H}_{ij}H_i + (h_{ij}^{\Omega} LH\mathcal{H}_{ij}H_i)^{\top} P \right). \end{aligned} \quad (3.93)$$

In addition to this, matrix  $Q$  satisfies:

$$Q \leq -\sigma P. \quad (3.94)$$

Hence, inequality (3.92) is rewritten as:

$$\dot{V}(\tilde{x}) \leq -\sigma V(\tilde{x}) + \delta \|\omega\|^2. \quad (3.95)$$

The incorporation of (3.88) into (3.95) yields:

$$\dot{V}(\tilde{x}) \leq -\sigma \lambda_{\max}(P) \|\tilde{x}\|^2 + \delta \|\omega\|^2. \quad (3.96)$$

One can easily infer that the conditions described in (3.88) and (3.96) are equivalent to (1.34) and (1.35), respectively. Hence, the function specified in (3.85) is an ISS-Lyapunov function along with

$$\alpha_1(\|\tilde{x}(t)\|) = \lambda_{\min}(P) \|\tilde{x}\|^2, \quad \alpha_2(\|\tilde{x}(t)\|) = \lambda_{\max}(P) \|\tilde{x}(t)\|^2$$

,

$$\alpha_3(\tilde{x}) = -\sigma \lambda_{\max}(P) \|\tilde{x}\|^2, \quad \text{and} \quad \alpha_4(\omega) = \delta \|\omega\|^2.$$

Since system (3.84) admits an ISS-Lyapunov function (3.85), from Theorem 1.3, it is ISS w.r.t.  $\omega$ . Thus, statement I of the theorem is proved.

### B. The proof of statements II and III:

The solution of (3.95) implies that the trajectories of  $V(\tilde{x}, t)$  satisfies:

$$\begin{aligned} V(\tilde{x}(t)) &\leq V(\tilde{x}_0)e^{-\sigma t} + \delta e^{-\sigma t} \int_0^t e^{\sigma s} \|\omega(s)\|_2^2 ds \\ &\leq V(\tilde{x}_0)e^{-\sigma t} + \delta \sup_{s \in [0, t]} \|\omega(s)\|_2^2 e^{-\sigma t} \int_0^t e^{\sigma s} ds \\ &\leq V(\tilde{x}_0)e^{-\sigma t} + \frac{\delta}{\sigma} (1 - e^{-\sigma t}) \sup_{s \in [0, t]} \|\omega(s)\|_2^2. \end{aligned} \quad (3.97)$$

Since  $0 \leq 1 - e^{-\sigma t} \leq 1$ ,  $\sup_{s \in [0, t]} \|\omega(s)\|_2^2 \leq \|\omega\|_{\mathcal{L}_\infty}^2$ ,

$$V(\tilde{x}(t)) \leq V(\tilde{x}_0)e^{-\sigma t} + \frac{\delta}{\sigma} \|\omega\|_{\mathcal{L}_\infty}^2. \quad (3.98)$$

By using (3.88) and (3.98), we obtain:

$$\|\tilde{x}(t)\|^2 \leq \frac{V(\tilde{x}, t)}{\lambda_{\min}(P)} \leq \frac{e^{-\sigma t} V(\tilde{x}, 0) + \delta \|\omega(\cdot)\|_\infty^2}{\lambda_{\min}(P)} \leq \frac{\lambda_{\max}(P) \|\tilde{x}_0\|^2 e^{-\sigma t} + \delta \sigma^{-1} \|\omega(\cdot)\|_\infty^2}{\lambda_{\min}(P)}. \quad (3.99)$$

It leads to:

$$\|\tilde{x}(t)\|^2 \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \|\tilde{x}_0\|^2 e^{-\sigma t} + \frac{\delta}{\sigma \lambda_{\min}(P)} \|\omega(\cdot)\|_\infty^2. \quad (3.100)$$

From (3.100), one can easily derive the inequality (3.86) which is in the form of (1.32).

Further, when  $t$  tends to  $\infty$ , (3.86) is modified as:  $\|\tilde{x}(\infty)\| \leq \sqrt{\frac{\delta}{\sigma \lambda_{\min}(P)}} \|\omega(\cdot)\|_\infty$ , which is in the form of (3.87).

Therefore, Statements II and III are proved.

This ends the proof. □

#### Remark 3.5

In the case of  $\omega = 0$ , the condition specified in (3.86) is rephrased as:

$$\|\tilde{x}(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} e^{-\frac{\sigma}{2}t} \|\tilde{x}_0\|.$$

Hence, the error dynamic (3.84) is exponentially stable in the absence of disturbances (at  $\omega = 0$ ).

### 3.3.3.2 Matrix-multipliers-based LMI formulation

This segment of the chapter is dedicated to the development of an LMI condition which is based on the conditions illustrated in Theorem 3.4 .

One of the essential conditions for the existence of ISS is depicted in (3.95). The inequality (3.95) is true if matrix  $Q$  holds (3.94). The condition illustrated in (3.94) is very beneficial for formulating the LMI condition.

From (3.93) and (3.94), we get:

$$PA_L + A_L^\top P + (PE_L)^\top (\delta \mathbb{I}) (PE_L) + \sum_{i,j=1}^{m,\bar{n}} \left( f_{ij}^\Omega PGF_{ij} F_i + (f_{ij}^\Omega GF_{ij} F_i)^\top P \right) - \sum_{i,j=1}^{r,\bar{p}} \left( h_{ij}^\Omega PLH\mathcal{H}_{ij} H_i + (h_{ij}^\Omega LH\mathcal{H}_{ij} H_i)^\top P \right) \leq -\sigma P. \quad (3.101)$$

Inequality (3.101) is true if

$$L_1 + \mathcal{N}_1 + \mathcal{N}_2 \leq 0, \quad (3.102)$$

where

$$L_1 = \begin{bmatrix} A^\top P + PA - R^\top C - C^\top R + \sigma P & PE - R^\top D \\ \star & -\delta \mathbb{I} \end{bmatrix}, \quad (3.103)$$

$$\mathcal{N}_1 = \left[ \sum_{i,j=1}^{m,\bar{n}} \left( \underbrace{\begin{bmatrix} PGF_{ij} \\ \mathbb{O} \end{bmatrix}}_{U_{ij}^\top} \underbrace{f_{ij}^\Omega \begin{bmatrix} F_i & \mathbb{O} \end{bmatrix}}_{F_i} + V_{ij}^\top U_{ij} \right) \right], \quad (3.104)$$

$$\mathcal{N}_2 = \left[ \sum_{i,j=1}^{r,\bar{p}} \left( \underbrace{\begin{bmatrix} -R^\top H\mathcal{H}_{ij} \\ \mathbb{O} \end{bmatrix}}_{M_{ij}^\top} \underbrace{h_{ij}^\Omega \begin{bmatrix} H_i & \mathbb{O} \end{bmatrix}}_{H_i} + N_{ij}^\top M_{ij} \right) \right]. \quad (3.105)$$

and  $R = L^\top P$ .

To avoid cumbersome equations, the subsequent notations are introduced:

$$U = \left[ U_{11}^\top \quad \dots \quad U_{1\bar{n}}^\top \quad \dots \quad U_{m1}^\top \quad \dots \quad U_{m\bar{n}}^\top \right]^\top, \quad (3.106)$$

$$V = \left[ V_{11}^\top \quad \dots \quad V_{1\bar{n}}^\top \quad \dots \quad V_{m1}^\top \quad \dots \quad V_{m\bar{n}}^\top \right]^\top, \quad (3.107)$$

$$M = \left[ M_{11}^\top \quad \dots \quad M_{1\bar{p}}^\top \quad \dots \quad M_{r1}^\top \quad \dots \quad M_{r\bar{p}}^\top \right]^\top, \quad (3.108)$$

$$N = \left[ N_{11}^\top \quad \dots \quad N_{1\bar{p}}^\top \quad \dots \quad N_{r1}^\top \quad \dots \quad N_{r\bar{p}}^\top \right]^\top, \quad (3.109)$$

where  $U_{ij}$ ,  $V_{ij}$ ,  $M_{ij}$  and  $N_{ij}$  are described in (3.104) and (3.105). Further,  $V$  and  $N$  are represented as follows:

$$V = F\Phi, \quad (3.110)$$

$$N = H\Psi, \quad (3.111)$$

where

$$\mathbb{F} = \text{block-diag}(\underbrace{\mathbb{F}_1, \dots, \mathbb{F}_1}_{\bar{n} \text{ times}}, \dots, \underbrace{\mathbb{F}_m, \dots, \mathbb{F}_m}_{\bar{n} \text{ times}}), \quad (3.112)$$

$$\Phi^\top = \begin{bmatrix} f_{11}^\Omega \mathbb{I} & \dots & f_{1\bar{n}}^\Omega \mathbb{I} & \dots & f_{m1}^\Omega \mathbb{I} & \dots & f_{m\bar{n}}^\Omega \mathbb{I} \end{bmatrix}, \quad (3.113)$$

$$\mathbb{H} = \text{block-diag}(\underbrace{\mathbb{H}_1, \dots, \mathbb{H}_1}_{\bar{p} \text{ times}}, \dots, \underbrace{\mathbb{H}_r, \dots, \mathbb{H}_r}_{\bar{p} \text{ times}}), \quad (3.114)$$

$$\Psi^\top = \begin{bmatrix} h_{11}^\Omega \mathbb{I} & \dots & h_{1\bar{p}}^\Omega \mathbb{I} & \dots & h_{r1}^\Omega \mathbb{I} & \dots & h_{r\bar{p}}^\Omega \mathbb{I} \end{bmatrix}. \quad (3.115)$$

Hence, we can express (3.104) and (3.105) in the ensuing form:

$$\mathcal{N}_1 = \mathbf{U}^\top (\mathbb{F}\Phi) + (\mathbb{F}\Phi)^\top \mathbf{U}, \quad (3.116)$$

$$\mathcal{N}_2 = \mathbf{M}^\top (\mathbb{H}\Psi) + (\mathbb{H}\Psi)^\top \mathbf{M}. \quad (3.117)$$

The following inequalities are derived by implementing (A.10) on (3.116) and (3.117):

$$\mathcal{N}_1 \leq \frac{1}{2} \left[ (\mathbf{U} + \mathbb{Z}\mathbb{F}\Phi)^\top \mathbb{Z}^{-1} (\mathbf{U} + \mathbb{Z}\mathbb{F}\Phi) \right], \quad (3.118)$$

$$\mathcal{N}_2 \leq \frac{1}{2} \left[ (\mathbf{M} + \mathbb{S}\mathbb{H}\Psi)^\top \mathbb{S}^{-1} (\mathbf{M} + \mathbb{S}\mathbb{H}\Psi) \right], \quad (3.119)$$

where matrices  $\mathbb{Z}$  and  $\mathbb{S}$  are defined in (2.82) and (2.83), respectively.

Thus, the inequality (3.101) is satisfied if

$$L_1 + \frac{1}{2} \left[ (\mathbf{U} + \mathbb{Z}\mathbb{F}\Phi)^\top \mathbb{Z}^{-1} (\mathbf{U} + \mathbb{Z}\mathbb{F}\Phi) \right] + \frac{1}{2} \left[ (\mathbf{M} + \mathbb{S}\mathbb{H}\Psi)^\top \mathbb{S}^{-1} (\mathbf{M} + \mathbb{S}\mathbb{H}\Psi) \right] \leq 0. \quad (3.120)$$

If we implement the steps from (3.47) to (3.52) described in Section 3.2.3, the ensuing inequality is deduced:

$$L_1 + \left[ \frac{1}{2} (\mathbf{U} + \mathbb{Z}\mathbb{F}\Phi)^\top \mathbb{Z}^{-1} (\mathbf{U} + \mathbb{Z}\mathbb{F}\Phi) \right]_{\forall \Phi \in \mathcal{F}_m} + \left[ \frac{1}{2} (\mathbf{M} + \mathbb{S}\mathbb{H}\Psi)^\top \mathbb{S}^{-1} (\mathbf{M} + \mathbb{S}\mathbb{H}\Psi) \right]_{\forall \Psi \in \mathcal{H}_r} \leq 0, \quad (3.121)$$

where  $\mathcal{F}_m$  and  $\mathcal{H}_r$  are bounded convex sets, and their sets of vertices are given by

$$\begin{aligned} \mathcal{V}_{\mathcal{F}_m} &= \{ \{F_{11}, \dots, F_{1\bar{n}}, \dots, F_{m1}, \dots, F_{m\bar{n}}\} : F_{ij} \in [0, f_{ij}^\Omega] \}, \\ \mathcal{V}_{\mathcal{H}_r} &= \{ \{H_{11}, \dots, H_{1\bar{p}}, \dots, H_{r1}, \dots, H_{r\bar{p}}\} : H_{ij} \in [0, h_{ij}^\Omega] \}. \end{aligned}$$

Now, we are ready to state the subsequent theorem.

### Theorem 3.5

Let us consider two symmetric positive definite matrices  $\mathbb{Z}$  and  $\mathbb{S}$ , which are defined in (2.82) and (2.83), respectively. If there exist matrices  $P > 0 \in \mathbf{S}^n$ ,  $R \in \mathbb{R}^{p \times n}$ , and positive scalars  $\delta, \sigma$  such that the following optimization problem is solvable:

$$\begin{aligned} &\min \delta \text{ subject to} \\ &\begin{bmatrix} \mathbb{L}_1 & (\mathbf{U} + \mathbb{Z}\mathbb{F}\Phi)^\top & (\mathbf{M} + \mathbb{S}\mathbb{H}\Psi)^\top \\ \star & -2\mathbb{Z} & \mathbb{O} \\ \star & \star & -2\mathbb{S} \end{bmatrix} < 0 \quad \forall \Phi \in \mathcal{F}_m, \forall \Psi \in \mathcal{H}_r, \end{aligned} \quad (3.122)$$

where  $L_1, \mathbb{U}, \mathbb{M}, \mathbb{F}, \mathbb{H}, \Phi$ , and  $\Psi$  are defined in (3.103), (3.106), (3.108), (3.112), (3.114), (3.113) and (3.115), respectively. Then the estimation error dynamic (3.84) is ISS w.r.t.  $\omega$ . Additionally,  $L = P^{-1}R^\top$  is used to determine the gain matrix  $L$ .

**Proof:** The Schur's compliment of (3.121) results in LMI (3.122). According to the convexity principal [18], if LMI (3.122) is solved for every elements of  $\Phi \in \mathcal{F}_{F_m}$  and  $\Psi \in \mathcal{G}_{G_r}$ , then the condition depicted in (3.95) is satisfied. It ensures that all conditions specified in Theorem 3.4 are fulfilled by the system (3.84). Hence, from Theorem 3.4, system (3.84) is ISS w.r.t.  $\omega$ . Thus, the proof is completed.  $\square$

### 3.3.4 Observer design for systems having nonlinear dynamics and linear outputs

The LMI proposed in Segment 3.3.3.2 is applicable to systems with nonlinear dynamics and nonlinear outputs. In this part, the aforementioned LMI is extended for a class of nonlinear systems having linear measurements. This type of system is illustrated as follows:

$$\begin{aligned} \dot{x} &= Ax + Gf(x) + Bu + E\omega, \\ y &= Cx + D\omega, \end{aligned} \quad (3.123)$$

where all variables and parameters remain consistent with those defined in (3.68). The nonlinear function  $f(x)$  admits the detailed form described in (3.69) and fulfils the Assumption 3.1. Similar to the new observer structure illustrated in (3.71), the following observer form is used for the state estimation of the system (3.123):

$$\dot{\hat{x}} = A\hat{x} + G(f \circ \pi_{\Omega_f}(\hat{x})) + Bu + L(y - C\hat{x}), \quad (3.124)$$

where all variables and parameters adhere to the definitions showcased in (3.71). Analogous to (3.75), the estimation error dynamic for the preceding observer (3.124) is given by,

$$\dot{\tilde{x}} = A_L\tilde{x} + G\Delta_f + E_L\omega, \quad (3.125)$$

where  $A_L$  and  $E_L$  are outlined in (3.76) and (3.77), respectively. Whereas,  $\Delta_f = f(x) - f \circ \pi_{\Omega_f}(\hat{x})$ .

The incorporation of (3.80) into (3.125) yields:

$$\dot{\tilde{x}} = A_L\tilde{x} + \sum_{i,j=1}^{m,\bar{n}} f_{ij}^\Omega G F_{ij} F_i \tilde{x} + E_L\omega. \quad (3.126)$$

The subsequent corollary portrays sufficient conditions to ensure that the system (3.126) is ISS w.r.t.  $\omega$ .

#### Corollary 3.3

Let us presume that there exist the matrices  $P = P^\top > 0 \in \mathbb{R}^{n \times n}$  and  $R \in \mathbb{R}^{p \times n}$ , scalars  $\delta, \sigma > 0$  along with matrices  $\mathbb{Z} = \mathbb{Z}^\top > 0$  and  $\mathbb{S} = \mathbb{S}^\top > 0$ , which are defined in (2.82) and (2.83), respectively. The error dynamic (3.126) is ISS w.r.t.  $\omega$  if the following optimization problem

is solvable:

$$\begin{aligned} & \min \delta \text{ subject to} \\ & \begin{bmatrix} \mathbb{L}_1 & (\mathbb{U} + \mathbb{Z}\mathbb{F}\Phi)^\top \\ \star & -2\mathbb{Z} \end{bmatrix} < 0 \quad \forall \Phi \in \mathcal{F}_m, \end{aligned} \quad (3.127)$$

where all the variables are specified in Theorem 3.5 . Moreover,  $L = P^{-1}R^\top$  is deployed to calculate the gain matrix  $L$ .

**Proof:** The proof is very straightforward. One can follow the steps showcased in the proof of Theorem 3.5 to obtain LMI (3.127). Further, through the utilisation of Theorem 3.5 , it is easy to infer that the system (3.126) is ISS w.r.t.  $\omega$  such that the trajectories of the solution of (3.126) satisfy (3.86) and (3.87).  $\square$

### 3.4 Analysing the effectiveness of matrix-multipliers-based LMIs

The introduction of the newly defined matrix multipliers aids in the improvement of the LMI conditions as compared to the existing approaches. These improvements are mainly because of the additional number of decision variables in the proposed LMIs. Hence, these matrix multipliers play a vital role in LMI enhancement. In addition to this, one must know how these matrix multipliers add extra numbers of decision variables. The objective of this section is to tackle such questions and to prove the uniqueness of the proposed LMIs. First, the computation of the number of decision variables inside the derived LMIs is presented. Further, the comparison with existing LMI approaches is provided to validate the novelty of the proposed method. In addition to this, the superiority of the proposed matrix multipliers over the different possible other matrix structures is highlighted.

#### 3.4.1 Number of decision variables

The utilisation of the matrices  $\mathbb{Z}$  and  $\mathbb{S}$  in LMIs (3.44) and (3.53) allow the inclusion of additional numbers of decision variables. Both LMIs include the following number of decision variables:

$$N_{dv_1} = \underbrace{np + \frac{n(n+1)}{2}}_{\mathcal{N}_P} + q + \underbrace{\left(\frac{m\bar{n}(m\bar{n}+1)}{2}\right)\left(\frac{\bar{n}(\bar{n}+1)}{2}\right)}_{\mathcal{N}_{\text{add}_1}} + \underbrace{\left(\frac{r\bar{p}(r\bar{p}+1)}{2}\right)\left(\frac{\bar{p}(\bar{p}+1)}{2}\right)}_{\mathcal{N}_{\text{add}_2}}, \quad (3.128)$$

where  $\mathcal{N}_P$ ,  $\mathcal{N}_{\text{add}_1}$  and  $\mathcal{N}_{\text{add}_2}$  are the number of variables obtained from matrices  $\mathcal{A}_L$ ,  $\mathbb{Z}$  and  $\mathbb{S}$ , respectively. Moreover, the terms  $\mathcal{N}_{\text{add}_1}$  and  $\mathcal{N}_{\text{add}_2}$  represent the additional number of decision variables in the proposed LMIs. The total number of additional variables is given by,

$$\mathcal{N}_{\text{add}} = \mathcal{N}_{\text{add}_1} + \mathcal{N}_{\text{add}_2} = \left(\frac{m\bar{n}(m\bar{n}+1)}{2}\right)\left(\frac{\bar{n}(\bar{n}+1)}{2}\right) + \left(\frac{r\bar{p}(r\bar{p}+1)}{2}\right)\left(\frac{\bar{p}(\bar{p}+1)}{2}\right) \quad (3.129)$$

Now, let us compute the number of decision variables in the subsequent cases:

1 Case 1:  $\mathbb{Z}$  and  $\mathbb{S}$  as the block-diagonal matrix multipliers (analogous to [14] and [22])

In this case,

$$\mathcal{N}_{\text{add}}^1 = m\bar{n}\left(\frac{\bar{n}(\bar{n}+1)}{2}\right) + r\bar{p}\left(\frac{\bar{p}(\bar{p}+1)}{2}\right). \quad (3.130)$$

2 Case 2:  $\mathbb{Z}$  and  $\mathbb{S}$  as the diagonal matrix multipliers (same as the one used in [13])

Here, the number of additional variables is achieved as follows:

$$\mathcal{N}_{\text{add}}^2 = m\bar{n}^2 + r\bar{p}^2. \quad (3.131)$$

Since  $m, \bar{n} \in \mathbb{Z}^+$ , that is,  $m$  and  $\bar{n}$  are positive integers, it is easy to interpret

$$\mathcal{N}_{\text{add}}^2 \leq \mathcal{N}_{\text{add}}^1 \leq \mathcal{N}_{\text{add}}. \quad (3.132)$$

Hence, the number of additional decision variables obtained from the proposed matrix multipliers is greater than the one obtained in the existing methods. These additional variables add extra degrees of freedom and may improve the feasibility of LMI.

### 3.4.2 A comparison of the number of additional decision variables in different matrix multiplier structures

The structure of the matrix  $\mathbb{Z}$  (2.82) is a generalised form of any symmetric positive definite matrix. However, there are other possible ways to represent a symmetric positive definite matrix. This part of the section emphasises the effectiveness of the used structure over other possible forms of  $\mathbb{Z}$ .

Multiple possible structures of  $\mathbb{Z} = \mathbb{Z}^\top > 0$  are described as follows:

I Structure 1 - A generalised form:

$$\mathbb{Z} = \begin{bmatrix} \mathbb{Z}_1 & \mathbb{Z}_{b_2^1} & \cdots & \mathbb{Z}_{b_m^1} \\ \star & \mathbb{Z}_2 & \cdots & \mathbb{Z}_{b_m^2} \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \cdots & \mathbb{Z}_m \end{bmatrix}, \text{ where } \mathbb{Z}_i = \begin{bmatrix} Z_{i1} & Z_{a_{i2}^1} & \cdots & Z_{a_{i\bar{n}}^1} \\ \star & Z_{i2} & \cdots & Z_{a_{i\bar{n}}^2} \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \cdots & Z_{i\bar{n}} \end{bmatrix} \forall i \in \{1, \dots, m\}, \quad (3.133)$$

$$\mathbb{Z}_{b_i^j} = \begin{bmatrix} Z_{b_{i1}^{j1}} & Z_{b_{i2}^{j1}} & \cdots & Z_{b_{i\bar{n}}^{j1}} \\ Z_{b_{i1}^{j2}} & Z_{b_{i2}^{j2}} & \cdots & Z_{b_{i\bar{n}}^{j2}} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{b_{i1}^{j\bar{n}}} & Z_{b_{i2}^{j\bar{n}}} & \cdots & Z_{b_{i\bar{n}}^{j\bar{n}}} \end{bmatrix} \forall i \in \{2, \dots, m\}, j \in \{1, \dots, m-1\},$$

where  $Z_{ij} = Z_{ij}^\top > 0 \in \mathbb{R}^{\bar{n} \times \bar{n}}$ ,  $Z_{a_{ij}^k} = Z_{a_{ij}^k}^\top \geq 0 \in \mathbb{R}^{\bar{n} \times \bar{n}} \forall i, k \in \{1, \dots, m\}, \& j \in \{1, \dots, \bar{n}\}$ ;

$Z_{b_{ij}^{kj}} = Z_{b_{ij}^{kj}}^\top \geq 0 \in \mathbb{R}^{\bar{n} \times \bar{n}}, \forall i \in \{2, \dots, m\}, k \in \{1, \dots, m-1\}, \& j \in \{1, \dots, \bar{n}\}$  such that  $\mathbb{Z} > 0$ .

The structure represented by (3.133) is identical to the one defined in (2.82). For ease of comprehension, (3.133) is utilised further.

II Structure 2 - In (3.133),  $\mathbb{Z}_{b_i}$  is written under the form of  $\mathbb{Z}_{b_i} = \mathbb{Z}_{b_i}^\top$ , that is,

$$\mathbb{Z} = \begin{bmatrix} \mathbb{Z}_1 & \mathbb{Z}_{b_2^1} & \dots & \mathbb{Z}_{b_m^1} \\ \star & \mathbb{Z}_2 & \dots & \mathbb{Z}_{b_m^2} \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \dots & \mathbb{Z}_m \end{bmatrix}, \text{ where } \mathbb{Z}_i = \begin{bmatrix} \mathbb{Z}_{i1} & \mathbb{Z}_{a_{i2}^1} & \dots & \mathbb{Z}_{a_{i\bar{n}}^1} \\ \star & \mathbb{Z}_{i2} & \dots & \mathbb{Z}_{a_{i\bar{n}}^2} \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \dots & \mathbb{Z}_{i\bar{n}} \end{bmatrix} \forall i \in \{1, \dots, m\},$$

$$\text{and } \mathbb{Z}_{b_i^j} = \begin{bmatrix} \mathbb{Z}_{b_{i1}^{j1}} & \mathbb{Z}_{b_{i2}^{j1}} & \dots & \mathbb{Z}_{b_{i\bar{n}}^{j1}} \\ \star & \mathbb{Z}_{b_{i2}^{j2}} & \dots & \mathbb{Z}_{b_{i\bar{n}}^{j2}} \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \dots & \mathbb{Z}_{b_{i\bar{n}}^{j\bar{n}}} \end{bmatrix} \forall i \in \{2, \dots, m\}, j \in \{1, \dots, m-1\},$$
(3.134)

where  $Z_{ij} = Z_{ij}^\top > 0 \in \mathbb{R}^{\bar{n} \times \bar{n}}$ ,  $Z_{a_{ij}^k} = Z_{a_{ij}^k}^\top \geq 0 \in \mathbb{R}^{\bar{n} \times \bar{n}} \forall i, k \in \{1, \dots, m\}, \& j \in \{1, \dots, \bar{n}\}$ ;  
 $Z_{b_{ij}^{kj}} = Z_{b_{ij}^{kj}}^\top \geq 0 \in \mathbb{R}^{\bar{n} \times \bar{n}}, \forall i \in \{2, \dots, m\}, k \in \{1, \dots, m-1\}, \& j \in \{1, \dots, \bar{n}\}$ .

III Structure 3 - In (3.134), same  $\mathbb{Z}_{b_i}$  is considered instead of different  $\mathbb{Z}_{b_i}, \forall i \in \{2, \dots, m\}$ , i.e.,

$$\mathbb{Z} = \begin{bmatrix} \mathbb{Z}_1 & \mathbb{Z}_{b_2} & \dots & \mathbb{Z}_{b_m} \\ \star & \mathbb{Z}_2 & \dots & \mathbb{Z}_{b_m} \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \dots & \mathbb{Z}_m \end{bmatrix}, \text{ where } \mathbb{Z}_i = \begin{bmatrix} \mathbb{Z}_{i1} & \mathbb{Z}_{a_{i2}^1} & \dots & \mathbb{Z}_{a_{i\bar{n}}^1} \\ \star & \mathbb{Z}_{i2} & \dots & \mathbb{Z}_{a_{i\bar{n}}^2} \\ \star & \star & \ddots & \vdots \\ \star & \star & \dots & \mathbb{Z}_{i\bar{n}} \end{bmatrix} \forall i \in \{1, \dots, m\},$$

$$\text{and } \mathbb{Z}_{b_i} = \begin{bmatrix} \mathbb{Z}_{b_{i1}^1} & \mathbb{Z}_{b_{i2}^1} & \dots & \mathbb{Z}_{b_{i\bar{n}}^1} \\ \star & \mathbb{Z}_{b_{i2}^2} & \dots & \mathbb{Z}_{b_{i\bar{n}}^2} \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \dots & \mathbb{Z}_{b_{i\bar{n}}^{\bar{n}}} \end{bmatrix} \forall i \in \{2, \dots, m\},$$
(3.135)

where  $Z_{ij} = Z_{ij}^\top > 0 \in \mathbb{R}^{\bar{n} \times \bar{n}}$ ,  $Z_{a_{ij}^k} = Z_{a_{ij}^k}^\top \geq 0 \in \mathbb{R}^{\bar{n} \times \bar{n}} \forall i, k \in \{1, \dots, m\}, \& j \in \{1, \dots, \bar{n}\}$ ;  
 $Z_{b_{ij}^j} = Z_{b_{ij}^j}^\top \geq 0 \in \mathbb{R}^{\bar{n} \times \bar{n}}, \forall i \in \{2, \dots, m\}, \& j \in \{1, \dots, \bar{n}\}$ .

IV Structure 4 - In (3.135), same  $\mathbb{Z}_{a_{ij}}$  is considered, i.e.,

$$\mathbb{Z} = \begin{bmatrix} \mathbb{Z}_1 & \mathbb{Z}_{b_2} & \dots & \mathbb{Z}_{b_m} \\ \star & \mathbb{Z}_2 & \dots & \mathbb{Z}_{b_m} \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \dots & \mathbb{Z}_m \end{bmatrix}, \text{ where } \mathbb{Z}_i = \begin{bmatrix} \mathbb{Z}_{i1} & \mathbb{Z}_{a_{i2}} & \dots & \mathbb{Z}_{a_{i\bar{n}}} \\ \star & \mathbb{Z}_{i2} & \dots & \mathbb{Z}_{a_{i\bar{n}}} \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \dots & \mathbb{Z}_{i\bar{n}} \end{bmatrix} \forall i \in \{1, \dots, m\},$$

$$\text{and } \mathbb{Z}_{b_i} = \begin{bmatrix} \mathbb{Z}_{b_{i1}^1} & \mathbb{Z}_{b_{i2}^1} & \dots & \mathbb{Z}_{b_{i\bar{n}}^1} \\ \star & \mathbb{Z}_{b_{i2}^2} & \dots & \mathbb{Z}_{b_{i\bar{n}}^2} \\ \vdots & \vdots & \ddots & \vdots \\ \star & \star & \dots & \mathbb{Z}_{b_{i\bar{n}}^{\bar{n}}} \end{bmatrix} \forall i \in \{2, \dots, m\},$$
(3.136)

where  $Z_{ij} = Z_{ij}^\top > 0 \in \mathbb{R}^{\bar{n} \times \bar{n}}$ ,  $Z_{a_{ij}} = Z_{a_{ij}}^\top \geq 0 \in \mathbb{R}^{\bar{n} \times \bar{n}} \forall i \in \{1, \dots, m\}, \& j \in \{1, \dots, \bar{n}\}$ ;  
 $Z_{b_{ij}^j} = Z_{b_{ij}^j}^\top \geq 0 \in \mathbb{R}^{\bar{n} \times \bar{n}}, \forall i \in \{2, \dots, m\}, \& j \in \{1, \dots, \bar{n}\}$ .

All these variants of matrix  $\mathbb{Z}$  encompass the different number of decision variables. In the previous segment, it is already highlighted that matrix multipliers enhance LMI feasibility by adding extra numbers of decision variables. Therefore, it is essential to determine the exact number of decision variables provided by each above-mentioned structures. The computation of the number of variables is emphasised in Table 3.1.

Table 3.1: A quantitative analysis of the variables inside different matrix configurations

Structure	Calculation of elements	Total number of variables
Structure 1	$\underbrace{(1 + 2 + \dots + \bar{n} + \dots + m\bar{n})}_{\text{Number of elements}} =$	$\left(\frac{m\bar{n}(m\bar{n}+1)}{2}\right) \left(\frac{\bar{n}(\bar{n}+1)}{2}\right)$
Structure 2	$Z_{ij}^k: m\bar{n}$	$\frac{m\bar{n}(\bar{n}-1)}{2} + \frac{m\bar{n}(m-1)(\bar{n}+1)}{4} + m\bar{n}$ $= \frac{m\bar{n}(\bar{n}+1)(m+1)}{4} \left(\frac{\bar{n}(\bar{n}+1)}{2}\right)$
	$Z_{b_{ij}^{kj}}: \left(\sum_{a=1}^{\bar{n}-1} a\right) \left(\sum_{b=1}^{m-1} b\right) = \frac{m\bar{n}(m-1)(\bar{n}+1)}{4}$	
Structure 3	$Z_{ij}^k: m\bar{n}$	$\frac{m\bar{n}(\bar{n}-1)}{2} + \frac{\bar{n}(m-1)(\bar{n}+1)}{2} + m\bar{n}$ $= \frac{\bar{n}(\bar{n}+1)(2m-1)}{2} \left(\frac{\bar{n}(\bar{n}+1)}{2}\right)$
	$Z_{a_{ij}^k}: \left(\sum_{a=1}^{\bar{n}-1} a\right) m = \frac{m\bar{n}(\bar{n}-1)}{2}$ $Z_{b_{ij}^j}: \left(\left(\sum_{a=1}^{\bar{n}} a\right)\right) (m-1) = \frac{\bar{n}(m-1)(\bar{n}+1)}{2}$	
Structure 4	$Z_{ij}^k: m\bar{n}$	$= (2m-1)(2\bar{n}-1) \left(\frac{\bar{n}(\bar{n}+1)}{2}\right)$
	$Z_{a_{ij}^k}: \left(\sum_{a=1}^{\bar{n}-1} 1\right) m = m(\bar{n}-1)$ $Z_{b_{ij}^j}: \left(\left(\sum_{a=1}^{\bar{n}-1} 2\right) + 1\right) (m-1) = (m-1)(2\bar{n}-1)$	

Further, the numbers of decision variables obtained from various matrix configurations are compared to highlight the effectiveness of the employed matrix form (Structure 1, (3.133)). Let us define

$$n_1 = \left(\frac{m\bar{n}(m\bar{n}+1)}{2}\right) \left(\frac{\bar{n}(\bar{n}+1)}{2}\right); \quad n_2 = \left(\frac{m\bar{n}(\bar{n}+1)(m+1)}{4}\right) \left(\frac{\bar{n}(\bar{n}+1)}{2}\right);$$

$$n_3 = \left(\frac{\bar{n}(\bar{n}+1)(2m-1)}{2}\right) \left(\frac{\bar{n}(\bar{n}+1)}{2}\right); \quad \text{and } n_4 = ((2m-1)(2\bar{n}-1)) \left(\frac{\bar{n}(\bar{n}+1)}{2}\right).$$

Now, consider

$$n_1 - n_2 = \left(\frac{m\bar{n}(m\bar{n}+1)}{2}\right) \left(\frac{\bar{n}(\bar{n}+1)}{2}\right) - \left(\frac{m\bar{n}(\bar{n}+1)(m+1)}{4}\right) \left(\frac{\bar{n}(\bar{n}+1)}{2}\right),$$

$$= \left[\frac{m\bar{n}(m\bar{n}+1)}{2} - \frac{m\bar{n}(\bar{n}+1)(m+1)}{4}\right] \left(\frac{\bar{n}(\bar{n}+1)}{2}\right),$$

$$= \left(\frac{m\bar{n}}{2}\right) \left(\frac{(m-1)(\bar{n}-1)}{2}\right) \left(\frac{\bar{n}(\bar{n}+1)}{2}\right).$$

Since  $m, \bar{n} \in \mathbb{Z}^+$ ,

$$n_1 - n_2 \geq 0. \tag{3.137}$$

Similarly,

$$\begin{aligned} n_2 - n_3 &= \left( \frac{m\bar{n}(\bar{n}+1)(m+1)}{4} \right) \left( \frac{\bar{n}(\bar{n}+1)}{2} \right) - \left( \frac{\bar{n}(\bar{n}+1)(2m-1)}{2} \right) \left( \frac{\bar{n}(\bar{n}+1)}{2} \right), \\ &= \left[ \frac{m\bar{n}(\bar{n}+1)(m+1)}{4} - \frac{\bar{n}(\bar{n}+1)(2m-1)}{2} \right] \left( \frac{\bar{n}(\bar{n}+1)}{2} \right), \\ &= \left( \frac{\bar{n}(\bar{n}+1)}{2} \right) \left( \frac{(m-1)(m-2)}{2} \right) \left( \frac{\bar{n}(\bar{n}+1)}{2} \right). \end{aligned}$$

In light of the fact that  $m, \bar{n} \in \mathbb{Z}^+$ ,

$$n_2 - n_3 \geq 0. \quad (3.138)$$

Analogously,

$$\begin{aligned} n_3 - n_4 &= \left( \frac{\bar{n}(\bar{n}+1)(2m-1)}{2} \right) \left( \frac{\bar{n}(\bar{n}+1)}{2} \right) - ((2m-1)(2\bar{n}-1)) \left( \frac{\bar{n}(\bar{n}+1)}{2} \right), \\ &= \left[ \frac{\bar{n}(\bar{n}+1)(2m-1)}{2} - (2m-1)(2\bar{n}-1) \right] \left( \frac{\bar{n}(\bar{n}+1)}{2} \right), \\ &= (2m-1) \left( \frac{(\bar{n}-1)(\bar{n}-2)}{2} \right) \left( \frac{\bar{n}(\bar{n}+1)}{2} \right). \end{aligned}$$

As  $m, \bar{n} \in \mathbb{Z}^+$ , we get:

$$n_3 - n_4 \geq 0. \quad (3.139)$$

From (3.137), (3.138) and (3.139), it is easy to ascertain that:

$$n_1 \geq n_2 \geq n_3 \geq n_4. \quad (3.140)$$

It implies that the matrix form defined in (3.133) provides a higher number of additional decision variables as compared to other forms. Hence, the LMIs obtained by using these matrix multipliers possess more decision variables than the ones obtained from other configurations of matrix multipliers. Therefore, the introduction of the proposed matrix multipliers enhances the LMI conditions.

In the subsequent section, the effectiveness of the proposed matrix multiplier-based LMI condition is highlighted through numerical examples.

## 3.5 Evaluation of the proposed LMIs and the observers

The primary objective of this section is to emphasise the effectiveness of the proposed matrix multiplier-based LMIs over the existing approaches through numerical examples. In addition to this, the developed LMI-based observer is deployed on the same examples to validate its performance.

### 3.5.1 Numerical Example 1

A numerical example is employed to showcase the efficacy of the developed LMIs (3.44) and (3.53). Let us consider a nonlinear system in the form of (3.2) with the ensuing parameters:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, D = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \text{ and } F = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Let us consider that both system dynamics and outputs are corrupted by the disturbance  $\omega \rightsquigarrow (0, 0.1)$ . Additionally, the system is governed by the input  $u = -0.1 \sin 10t$ .

The nonlinearities are described as follows:

$$f(x) = \begin{bmatrix} f_1(H_1x) \\ f_2(H_2x) \end{bmatrix} = \begin{bmatrix} \sin(\theta x_1) \\ \sin(\theta x_2) \end{bmatrix} \text{ and } g(F_1x) = \sin(\lambda x_1) \sin(\lambda x_2),$$

where  $H_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ ,  $H_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$  and  $F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ . Therefore,  $m = 2$ ,  $\bar{n} = 2$ ,  $r = 1$  and  $\bar{p} = 2$ .

In addition to this, the partial derivatives of  $f(\cdot)$  and  $g(\cdot)$  hold:

$$-\theta \leq \frac{\partial f_1}{\partial x_1} \leq \theta; \quad -\theta \leq \frac{\partial f_2}{\partial x_2} \leq \theta; \quad -\lambda \leq \frac{\partial g}{\partial x_1} \leq \lambda; \quad \text{and} \quad -\lambda \leq \frac{\partial g}{\partial x_2} \leq \lambda.$$

It leads to:  $f_{a_{11}} = f_{a_{21}} = -\theta$ ;  $f_{b_{11}} = f_{b_{21}} = \theta$ ;  $g_{a_{11}} = g_{a_{12}} = -\lambda$ ;  $g_{b_{11}} = g_{b_{12}} = \lambda$ .

With reference to Remark 3.2, one can obtain:

$$\tilde{f}_{a_{11}} = \tilde{f}_{a_{21}} = 0; \quad \tilde{f}_{b_{11}} = \tilde{f}_{b_{21}} = 2\theta; \quad \tilde{g}_{a_{11}} = \tilde{g}_{a_{12}} = 0; \quad \tilde{g}_{b_{11}} = \tilde{g}_{b_{12}} = 2\lambda.$$

In addition to this,  $\tilde{A} = A + f_{a_{11}}G\mathcal{H}_{11}H_1 + f_{a_{21}}G\mathcal{H}_{21}H_2 - g_{a_{11}}LFG_{11}G_1 - g_{a_{12}}LFG_{12}G_1$  is used instead of  $A$  in the term  $\mathcal{A}_L$  of LMIs (3.44) and (3.53).

The performance of LMIs is evaluated in the following cases:

1. **Case 1:** LMI (3.53) with the proposed matrices,

$$\mathbb{Z} = \begin{bmatrix} Z_{11} & Z_{b_{21}} \\ Z_{b_{21}} & Z_{21} \end{bmatrix}; \quad \mathbb{S} = \begin{bmatrix} S_{11} & S_{a_{12}} \\ S_{a_{12}} & S_{12} \end{bmatrix} \quad (3.141)$$

where  $Z_{11}, Z_{21}, Z_{b_{21}} \in \mathbb{R}^{\bar{n} \times \bar{n}}$ , and  $S_{11}, S_{12}, S_{a_{12}} \in \mathbb{R}^{\bar{p} \times \bar{p}}$  are symmetric matrices such that  $\mathbb{Z}, \mathbb{S} > 0$ .

2. **Case 2:** LMI (3.44) with matrices defined in (3.141).

3. **Case 3:** LMI (3.44) along with

$$\mathbb{Z} = \begin{bmatrix} Z_{11} & \alpha Z_{21} \\ \alpha Z_{21} & Z_{21} \end{bmatrix}; \quad \mathbb{S} = \begin{bmatrix} S_{11} & \beta S_{12} \\ \beta S_{12} & S_{12} \end{bmatrix}, \quad (3.142)$$

where  $Z_{11} = Z_{11}^\top > 0 \in \mathbb{R}^{\bar{n} \times \bar{n}}$ ,  $Z_{21} = Z_{21}^\top > 0 \in \mathbb{R}^{\bar{n} \times \bar{n}}$ ,  $S_{11} = S_{11}^\top > 0 \in \mathbb{R}^{\bar{p} \times \bar{p}}$ ,  $S_{12} = S_{12}^\top > 0 \in \mathbb{R}^{\bar{p} \times \bar{p}}$ , and  $\alpha = 0.25, \beta = 0.25$  (similar to the one defined in Section 2.4).

4. **Case 4:** LMI (3.44) accompanied by

$$\mathbb{Z} = \text{block-diag}(Z_{11}, Z_{21}); \quad \mathbb{S} = \text{block-diag}(S_{11}, S_{12}), \quad (3.143)$$

where  $Z_{11}, Z_{21} \in \mathbb{R}^{\bar{n} \times \bar{n}}$  and  $S_{11}, S_{12} \in \mathbb{R}^{\bar{p} \times \bar{p}}$  are symmetric positive definite matrices.

5. **Case 5:** LMI (3.44) with

$$\mathbb{Z} = \text{block-diag}(Z_{11}, Z_{21}); \quad \mathbb{S} = \text{block-diag}(S_{11}, S_{12}), \quad (3.144)$$

where  $Z_{11} = \delta_1 \mathbb{I}_{\bar{n}}$ ,  $Z_{21} = \delta_2 \mathbb{I}_{\bar{n}}$ ,  $S_{11} = \delta_3 \mathbb{I}_{\bar{p}}$ ;  $S_{12} = \delta_4 \mathbb{I}_{\bar{p}}$  so that  $\mathbb{Z}, \mathbb{S} > 0$ .

Table 3.2: Optimal values of  $\sqrt{\mu}$  for different cases

No.		Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
1	$\theta = 0.2$ and $\lambda = 0.3$	1.3259	2.1704	5.5342	2.5007	Inf	1.3737
2	$\theta = 0.5$ and $\lambda = 0.25$	1.6874	5.0705	7.3810	7.3785	Inf	1.9786
3	$\theta = 0.9$ and $\lambda = 0.25$	1.9779	6.3298	7.6233	7.6192	Inf	2.0380
4	$\theta = 0.8$ and $\lambda = 0.8$	4.2978	9.5368	inf	inf	inf	9.6790
5	$\theta = 0.7$ and $\lambda = 1.2$	3.0463	8.3907	inf	inf	inf	10.6595
6	$\theta = 0.8$ and $\lambda = 1.5$	5.7551	10.1047	inf	inf	inf	33.8465
7	$\theta = 0.9$ and $\lambda = 2$	9.0558	10.6686	inf	inf	inf	63.4416

### 6. Case 6: LMI approach proposed in [14]

For each of the aforementioned cases, LMI is solved in the MATLAB toolbox, and the obtained optimal value of  $\sqrt{\mu}$  is recapitulated in Table 3.2.

Table 3.2 interprets that the LMI (3.53) provides a better optimal solution than other cases. It is due to the fact that the optimal value obtained in Case 1 is the smallest as compared to the same obtained in other cases. In certain scenarios, the solution of LMI (3.44) is better than the one estimated in Case 6 (i.e., [14]). All these results emphasise the efficiency of the matrix multiplier-based LMI approach over the existing methods. The proposed LMI conditions provide better feasibility than the existing ones. It is a consequence of the introduction of the newly defined matrix multipliers which adds some additional numbers of decision variables inside LMIs, and incorporates extra degrees of freedom. Thus, it leads to LMI enhancement.

The observer specified in (3.3) is deployed on the aforementioned example for the purpose of state estimation. In order to emphasise the performance of the proposed LMI-based observer, Case 1, Case 2, and Case 6 are taken into account for  $\theta = 0.7$  and  $\lambda = 1.2$ . In each case, LMI is solved in the MATLAB toolbox, and the gain matrix is computed. All these matrices are outlined in Table 3.3.

Table 3.3: Values of gain matrix  $L$  for different cases

	Case 1	Case 2	Case 6
$\theta = 0.7$ and $\lambda = 1.2$	$L = \begin{bmatrix} -0.0006 & 46.5548 \\ -0.0007 & 46.8323 \end{bmatrix}$	$L = \begin{bmatrix} -0.0000 & 335.7587 \\ -0.0000 & 392.2913 \end{bmatrix}$	$L = \begin{bmatrix} -0.0004 & 249.6421 \\ -0.0001 & 66.8295 \end{bmatrix}$

Further, the proposed observer (3.3) is implemented in MATLAB for each case by using all these gain matrices. The subsequent Figures 3.3, 3.4 and 3.5 depict the graphical representation of the estimation error in Case 1, Case 2, and Case 3, respectively.

The obtained value of  $\sqrt{\mu}$  in Case 1 is lesser than the same achieved in Case 2 and Case 3. Thus, Figure 3.3 showcase better noise compensation as compared to Figure 3.4 and Figure 3.5. In addition to this, all these Figures portray the asymptotic convergence of estimation error dynamics. Thus, observer performance is validated.

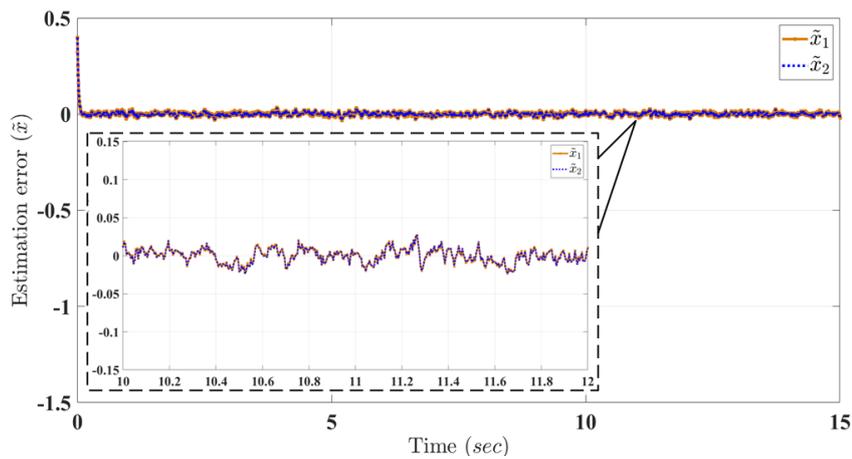


Figure 3.3: Plot of estimation error dynamic ( $\tilde{x}$ ) in Case 1

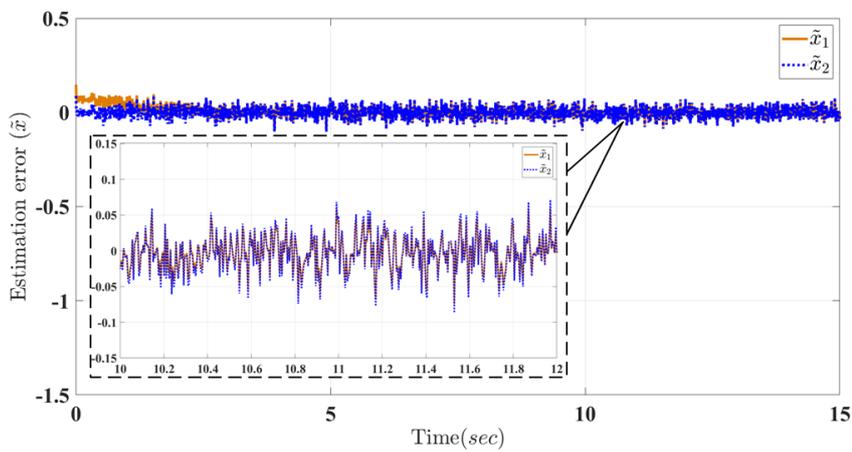


Figure 3.4: Plot of estimation error dynamic ( $\tilde{x}$ ) in Case 2

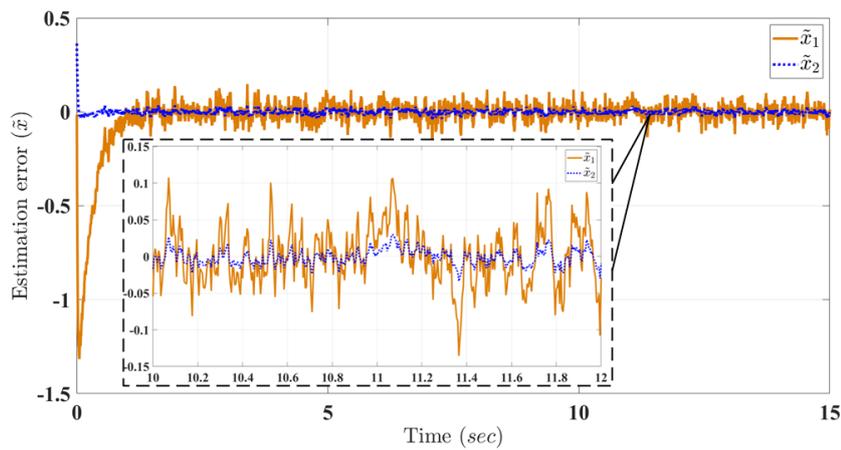


Figure 3.5: Plot of estimation error dynamic ( $\tilde{x}$ ) in Case 3

### 3.5.2 Numerical Example 2

Let us consider the nonlinear system (3.68) with the following parameters:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -5 \end{bmatrix}, G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, E = \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and} \\ D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The nonlinearities inside the system are illustrated as:  $f_1(F_1x) = x_2^2$ ,  $f_2(F_2x) = x_2x_3$ ,  $h_1(H_1x) = \sin x_1$  and  $h_2(H_2x) = \sin x_3$  along with  $F_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ,  $F_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ,

$$H_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \text{ and } H_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}. \text{ Hence, } m = 2, r = 2, \bar{n} = 3, \bar{p} = 3.$$

For initial condition  $x(0) = [-1 \quad -1 \quad -1]^\top$  and input  $u = \text{square}(2t)$ , the states are bounded in the following sets:

$$\begin{aligned} x \in \{ & -10.2929 \leq x_1 \leq -1, \\ & -1.0739 \leq x_2 \leq 0.2935, \\ & \text{and } -1 \leq x_3 \leq 0.4686 \} \forall t \in [0, 30]. \end{aligned}$$

In order to design an observer specified in (3.71) for the aforementioned system, the subsequent projection of the estimated states are considered:

$$f \circ \pi_{\Omega_f}(\hat{x}) = \begin{bmatrix} f_1 \circ \pi_{\Omega_{f_1}}(F_1\hat{x}) \\ f_2 \circ \pi_{\Omega_{f_2}}(F_2\hat{x}) \end{bmatrix}; \quad h \circ \pi_{\Omega_h}(\hat{x}) = \begin{bmatrix} h_1 \circ \pi_{\Omega_{h_1}}(H_1\hat{x}) \\ h_2 \circ \pi_{\Omega_{h_2}}(H_2\hat{x}) \end{bmatrix}.$$

As stated in the previous section,  $f_i(\pi_{\Omega_{f_i}}(F_i\hat{x})) = f_i(F_i\hat{x})$  for  $i \in \{1, 2\}$  and  $h_i(\pi_{\Omega_{h_i}}(H_i\hat{x})) = h_i(H_i\hat{x})$  for  $i \in \{1, 2\}$ . Thus, it ensures that projection on the estimated states does not alter its behaviour. In addition to this, it guarantees that the reconstructed states remain bounded in the above-defined sets as actual states.

The partial derivatives of  $f$  and  $h$  satisfy the following inequality for the considered duration:

$$\begin{aligned} -2 \times (-1.0739) \leq \frac{\partial f_1}{\partial x_2} \leq 2 \times (0.2935); \quad -1 \leq \frac{\partial f_2}{\partial x_2} \leq 0.5; \quad -1.0739 \leq \frac{\partial f_2}{\partial x_3} \leq 0.3; \\ -1.0739 \leq \frac{\partial h_1}{\partial x_1} \leq 1; \quad -1 \leq \frac{\partial h_2}{\partial x_3} \leq 0.2935. \end{aligned} \quad (3.145)$$

If one follow the steps described in Remark 3.2, it is easy to show that the nonlinear functions  $f_{ij}^{\Omega_{f_i}}$  and  $h_{ij}^{\Omega_{h_i}}$  ( $i = \{1, 2\}$ ) fulfil (3.82) and (3.83), respectively.

Further, the feasibility of LMI (3.122) is tested in the ensuing cases:

1) **Case 1:** LMI (3.122) with the succeeding matrices:

$$\mathbb{Z} = \begin{bmatrix} Z_{11} & Z_{b_{21}} & Z_{b_{22}} \\ Z_{b_{21}} & Z_{21} & Z_{a_{22}} \\ Z_{b_{22}} & Z_{a_{22}} & Z_{22} \end{bmatrix}; \quad \mathbb{S} = \begin{bmatrix} S_{11} & S_{b_{21}} \\ S_{b_{21}} & S_{21} \end{bmatrix}, \quad (3.146)$$

where  $Z_{ij}, Z_{b_{ij}}, Z_{a_{ij}} \in \mathbb{R}^{\bar{n} \times \bar{n}}$ ,  $S_{ij}, S_{b_{ij}} \in \mathbb{R}^{\bar{p} \times \bar{p}}, \forall i, j \in \{1, 2\}$  are symmetric matrices such that  $\mathbb{Z} > 0$  and  $\mathbb{S} > 0$ .

- 2) **Case 2:** LMI (3.122) with the following matrices which are similar to the one proposed in Section 2.4:

$$\mathbb{Z} = \begin{bmatrix} Z_{11} & \alpha Z_{21} & \alpha Z_{22} \\ \alpha Z_{21} & Z_{21} & \alpha Z_{22} \\ \alpha Z_{22} & \alpha Z_{22} & Z_{22} \end{bmatrix}; \mathbb{S} = \begin{bmatrix} S_{11} & \beta S_{21} \\ \beta S_{21} & S_{21} \end{bmatrix}, \quad (3.147)$$

where  $\alpha, \beta = 0.5$ , and all other matrices are defined in (3.146).

- 3) **Case 3:** LMI (3.122) accompanied by the block-diagonal matrices (same as in [14]), i.e.,

$$\begin{aligned} \mathbb{Z} &= \text{block-diag}(Z_{11}, Z_{21}, Z_{22}); \\ \mathbb{S} &= \text{block-diag}(S_{11}, S_{21}), \end{aligned} \quad (3.148)$$

where all matrices are described in (3.146).

- 4) **Case 4:** LMI (3.122) with identity matrices (same as in [12]), i.e.,

$$\begin{aligned} \mathbb{Z} &= \text{block-diag}(\mathbb{I}_{\bar{n}}, \mathbb{I}_{\bar{n}}, \mathbb{I}_{\bar{n}}) \\ \mathbb{S} &= \text{block-diag}(\mathbb{I}_{\bar{p}}, \mathbb{I}_{\bar{p}}). \end{aligned} \quad (3.149)$$

All LMIs defined in the above cases are solved by using the MATLAB toolbox. The estimated optimal values of  $\delta$ ,  $\gamma$  and gain matrix  $L$  are outlined in Table 3.4.

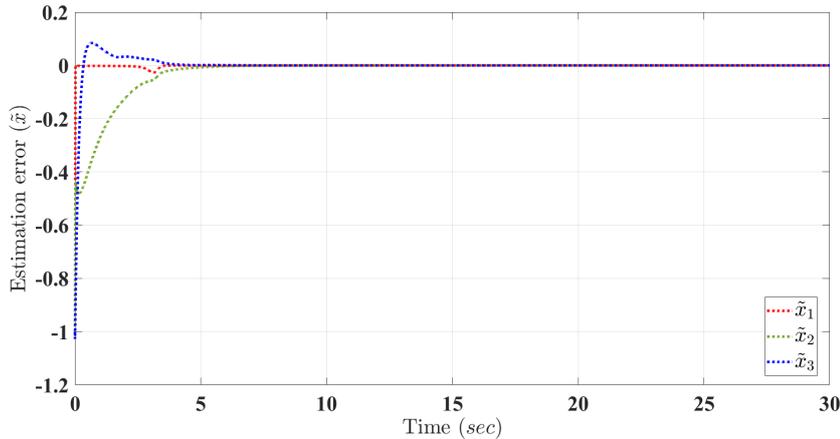


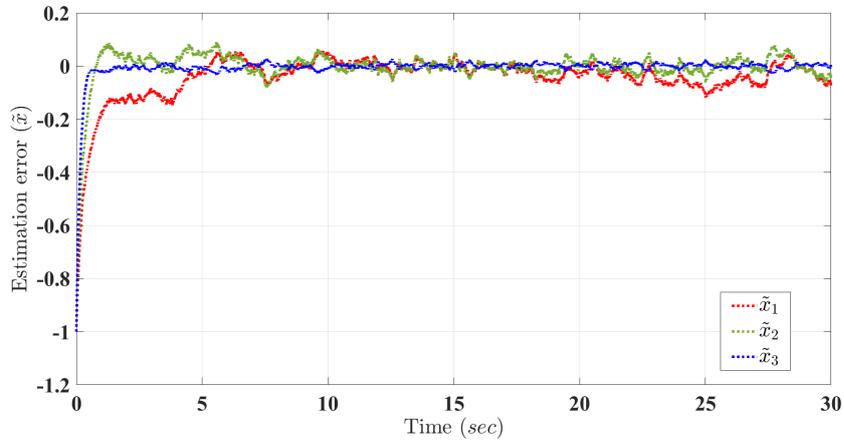
Figure 3.6: Plot of the estimation error ( $\tilde{x}$ ) in **Case 1**

Table 3.4 infers that the LMI (3.122) provides the best ISS gain with the proposed matrix multipliers (i.e., **Case 1**) as compared to other cases. The matrices defined in (3.147), (3.148), (3.149) are particular forms of the matrices stated in (3.146). Thus, the matrices of (3.146) encompasses more number of variables than the one shown in (3.147), (3.148), (3.149). Hence, it is obvious that the solution provided by the matrices of (3.146) is more general than other cases. These additional decision variables add some extra degrees of freedom from a feasibility point of view. Therefore, the inclusion of general matrix multipliers relaxes the existing LMI conditions from a feasibility point of view and noise compensation.

Let us consider that system dynamics and measurements are corrupted with the Gaussian noise ( $\omega \rightsquigarrow (0, 1)$ ). Further, the proposed observer (3.71) is deployed in a MATLAB environment to evaluate the observer performance. For each aforementioned case, the estimation error dynamics of the observer (3.71) is evaluated in MATLAB. Figures 3.6, 3.7 and 3.8 depict the plots of the

Table 3.4: Compiling the optimal values obtained from LMI solution

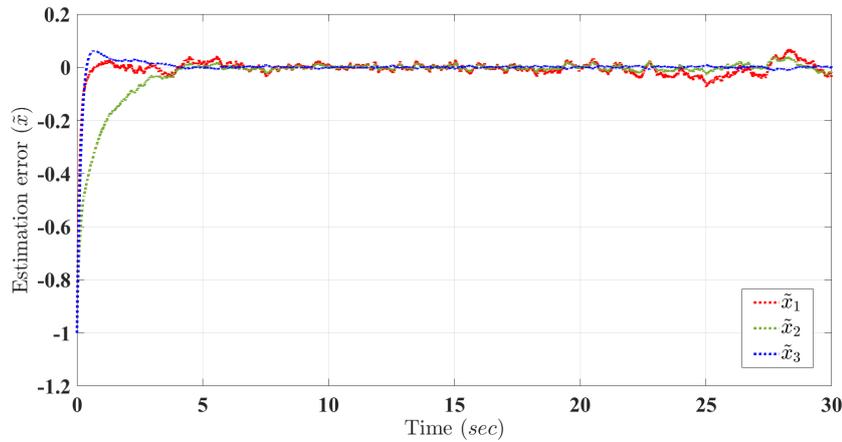
Parameters	Case 1	Case 2	Case 3	Case 4
$\delta$	$5.5499 \times 10^{-18}$	0.6160	0.4237	Infeasible
$\gamma = \sqrt{\frac{\delta}{\rho \lambda_{\min}(P)}}$	$3.0326 \times 10^{-12}$	2.4819	2.0585	
Gain matrix $L$	$\begin{bmatrix} 260.7229 & 0.0000 \\ 149.7926 & 0 \\ -17.8321 & 0.5000 \end{bmatrix}$	$\begin{bmatrix} 0.8643 & 1.4132 \\ 0.9302 & 1.7084 \\ -0.2009 & -0.3696 \end{bmatrix}$	$\begin{bmatrix} 4.4501 & 1.1749 \\ 2.4572 & 0.7608 \\ -0.6503 & 0.1732 \end{bmatrix}$	

Figure 3.7: Plot of the estimation error ( $\tilde{x}$ ) in Case 2

estimation errors in Case 1, Case 2 and Case 3, respectively. These figures showcase that the noise attenuation achieved in Case 1 is better than the one obtained in other cases. In addition to this, these figures highlight the asymptotic convergence of estimation errors. Further, the RMSE values of estimation errors are summarised in Table 3.5. It implies the same fact that the noise compensation is optimal in Case 1 as compared to the other cases. Thus, the performance of the proposed observer is validated.

Table 3.5: Comparison of RMSE values of  $\tilde{x}$  in different cases

Time		Case 1	Case 2	Case 3
$15 \leq t \leq 30$	$\tilde{x}_1$	$2.1450 \times 10^{-9}$	0.0454	0.0252
	$\tilde{x}_2$	$5.5216 \times 10^{-7}$	0.0254	0.0126
	$\tilde{x}_3$	$5.5216 \times 10^{-7}$	0.0254	0.0126

Figure 3.8: Plot of the estimation error ( $\tilde{x}$ ) in **Case 3**

### 3.6 Conclusion

This chapter presented two new LMI techniques for the design of a nonlinear observer. The primary component of the proposed methods is a generalized matrix multiplier, which leads to the inclusion of additional numbers of decision variables. Such extra variables provide some extra degrees of freedom, thus enhancing the feasibility of LMI. Further, the proposed novel matrix multiplier technique is combined with the LPV-based approach, which improves the feasibility of the LMI condition (3.44). In addition to this, a novel observer approach for the class of non-global Lipschitz nonlinear systems is developed. A well-known Hilbert transformation is employed in the observer structure to handle the local Lipschitz nonlinearities. Further, the asymptotic convergence of the estimation error is achieved through the utilisation of an ISS notion. The introduction of newly defined matrix multipliers allows the inclusion of additional decision variables as compared to the methods proposed in the literature. These extra variables relax the LMI conditions from a feasibility point of view by adding some extra degrees of freedom. The superiority of the proposed methods over the existing approaches is demonstrated through the numerical examples.

---

CHAPTER 4

# Observer-based stabilisation of the nonlinear systems: Exploiting the new matrix multiplier-based LMIs approach

## Contents

---

4.1	Introduction	93
4.2	Contextualising the problem statement	94
4.3	A robust circle-criterion based approach	97
4.4	ISS-based LMI formulation	101
4.5	Validating the performance of the developed methodology	104
4.5.1	Example 1: A single-link flexible joint robot model	104
4.5.2	Example 2: A numerical example of unstable system	106
4.6	Conclusion	109

---

## 4.1 Introduction

Over the last two decades, the topic of output-feedback stabilisation has garnered a lot of attention in control system engineering [89], [90]. This is due to external disturbances or noise, which can occur through sensors and/or actuators, and it induces instability in the system. The stability of such kind of system can be achieved through the employment of a controller. However, in many applications, obtaining the necessary state information for the development of the controller is not easily achievable due to economic or physical constraints on sensors. Through the utilisation of observers, it is easy to obtain the current knowledge of the system's state, regardless of whether it is measurable or not. Thus, observers play a pivotal role in the stabilisation problems. Several applications of observer-based controllers are showcased in [91], [69] and [26].

The design of observer-based controllers for linear systems is widely investigated by researchers, for example, [92], [16], [93], and so on. However, the nonlinear observer-based stabilisation task is a challenging task. The authors of [94] proposed the high-gain observer-based stabilisation methodology, whereas the disturbance-observer approach is utilised in [25]. Due to the advancements in computational technology over the last two decades, Linear Matrix Inequalities (LMIs) have garnered a lot of attention in control system engineering. The authors of [24], [26] and [90] incorporated LMIs in the development of nonlinear observer-based controllers. Though all these cited articles provide less conservative LMI conditions, there is a scope for further enhancements. In this chapter, the authors have developed the observer-based controller for the disturbance-affected nonlinear system using the previously proposed new-matrix multiplier-based LMI approach. As stated earlier, the derived LMIs are formulated using a newly defined matrix multiplier, which results in the inclusion of additional decision variables compared to the existing LMI methods. These variables add extra degrees of freedom and improve LMI feasibility. Further, the effectiveness of the proposed matrix multipliers is investigated in MATLAB using numerical examples.

The remainder of the chapter is organised as follows: Section 4.2 focuses on the formulation of the problem statement. For the proposed observer-based controller, a novel LMI condition is formulated in Section 4.3 using the  $\mathcal{H}_\infty$  criterion. Further, the property of ISS is utilised in Section 4.4 for deriving an LMI criterion. The performance of the proposed approaches is validated in Section 4.5 through numerical examples. In the last, Section 4.6 provides some concluding remarks.

## 4.2 Contextualising the problem statement

This section addresses the objective of the article. For better comprehensibility, a nonlinear system with nonlinear outputs under the effect of noise is considered for stabilisation purposes, and it is illustrated as follows:

$$\begin{aligned}\dot{x} &= Ax + Ff(x) + Bu + E\omega, \\ y &= Cx + Gg(x) + D\omega,\end{aligned}\tag{4.1}$$

where  $x \in \mathbb{R}^n$ ,  $y \in \mathbb{R}^p$  and  $u \in \mathbb{R}^s$  represent the states, the measurements of the systems, and the input of the system, respectively.  $\omega \in \mathbb{R}^q$  is the exogenous signal such as noise or disturbances affecting the system dynamics and outputs.  $A, G, B, C, F, E$  and  $D$  are known constant matrices of appropriate dimensions. The nonlinearities  $f(\cdot)$  and  $g(\cdot)$  are assumed to be globally Lipschitz, and admit detailed form (2.34) and (2.58), respectively. The ensuing observer-based controller is employed in this chapter for the stabilisation of (4.1):

$$\dot{\hat{x}} = A\hat{x} + Ff(\hat{x}) + Bu + L \left[ y - \left( C\hat{x} + Gg(\hat{x}) \right) \right],\tag{4.2a}$$

$$u = -K\hat{x},\tag{4.2b}$$

where  $\hat{x}$  denotes the estimated states, and matrices  $L$ ,  $K$  are the parameters of the observer-based controller. The depiction of observer-based stabilisation of the nonlinear system (4.1) methodology is showcased in Figure 4.1.

The estimation error of the observer-based controller (4.2) is defined as follows:

$$e = x - \hat{x}.$$

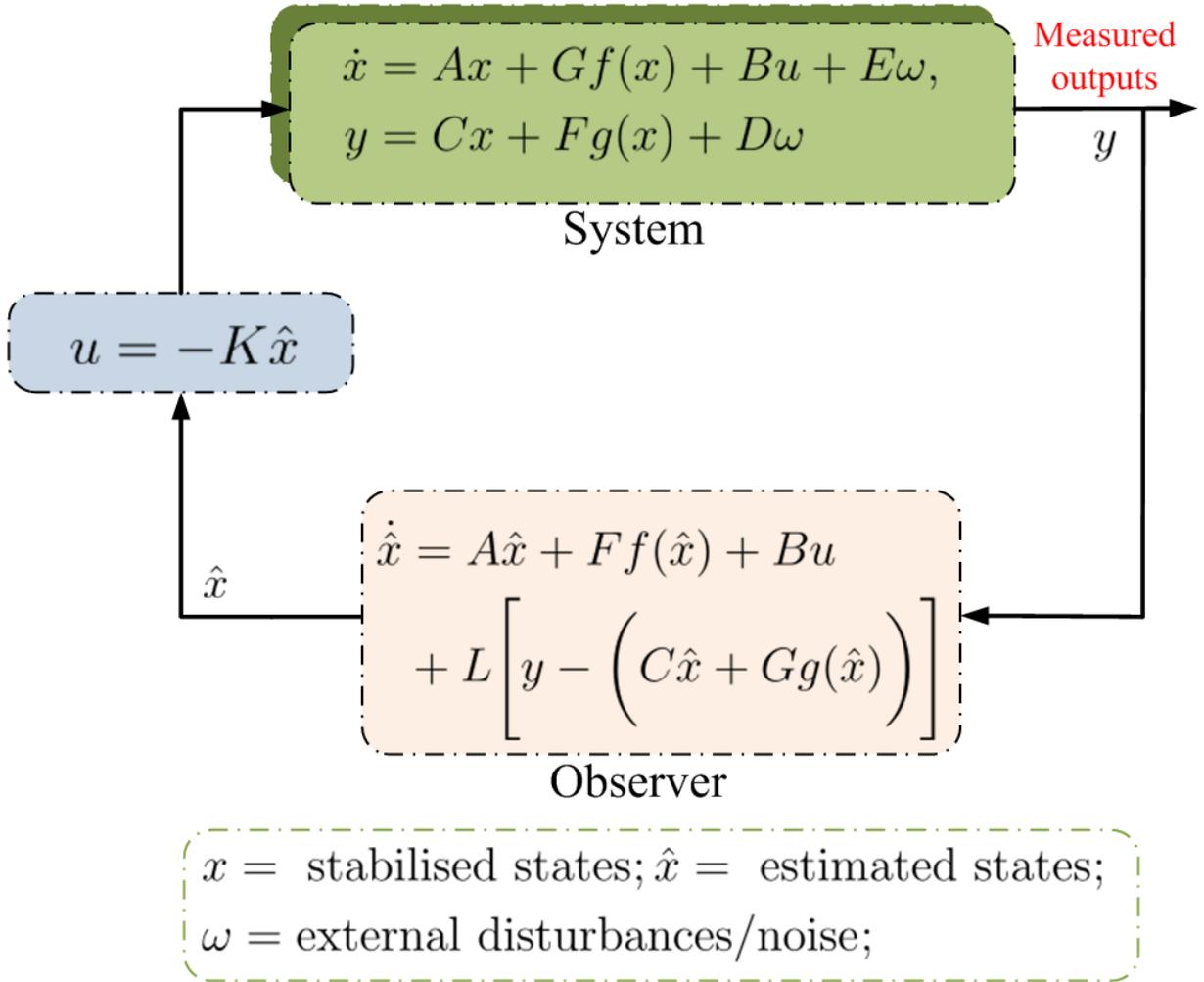


Figure 4.1: Illustration of observer-based controller

Thus, the dynamics of the estimation error ( $e$ ) are given by

$$\dot{e} = (A - LC)e + F \underbrace{(f(x) - f(\hat{x}))}_{\tilde{f}(x, \hat{x})} - LG \underbrace{(g(x) - g(\hat{x}))}_{\tilde{g}(x, \hat{x})} + (E - LD)\omega. \quad (4.3)$$

Analogous to (3.9) and (3.10), one can express  $\tilde{f}(x, \hat{x})$  and  $\tilde{g}(x, \hat{x})$  in the following manner:

$$\tilde{f}(x, \hat{x}) = \sum_{i,j=1}^{m, \bar{n}} f_{ij} \mathcal{H}_{ij} H_i e, \quad (4.4)$$

$$\tilde{g}(x, \hat{x}) = \sum_{i,j=1}^{r, \bar{p}} g_{ij} \mathcal{G}_{ij} G_i e, \quad (4.5)$$

where all the variables remain the same as the one specified in (3.9) and (3.10). The functions  $f_{ij}$  and  $g_{ij}$  satisfy (3.13) and (3.14), respectively.

Further, the incorporation of (4.3) and (4.5) in the dynamic (4.3) yields:

$$\dot{e} = (A - LC)e + \sum_{i,j=1}^{m, \bar{n}} f_{ij} F \mathcal{H}_{ij} H_i e - \sum_{i,j=1}^{r, \bar{p}} g_{ij} LG \mathcal{G}_{ij} G_i e + (E - LD)\omega. \quad (4.6)$$

Through the utilisation of (4.1) and (4.2b), one can obtain:

$$\dot{x} = (A - BK)x + BKe + Ff(x) + E\omega. \quad (4.7)$$

Similar to (4.4),  $f(x)$  is written as:

$$f(x) = f(x) - f(0) = \tilde{f}(x, 0) = \sum_{i,j=1}^{m,\bar{n}} \bar{f}_{ij} \mathcal{H}_{ij} H_i x, \quad (4.8)$$

with  $\bar{f}_{ij} \triangleq \bar{f}_{ij}(\theta_i^{0_{i,j-1}}, \theta_i^{0_{i,j}})$  and it holds

$$0 \leq \bar{f}_{ij} \leq f_{bij}. \quad (4.9)$$

Hence,

$$\dot{x} = (A - BK)x + BKe + \sum_{i,j=1}^{m,\bar{n}} \bar{f}_{ij} F \mathcal{H}_{ij} H_i x + E\omega. \quad (4.10)$$

Let us introduce an augmented vector  $\zeta = \begin{bmatrix} e \\ x \end{bmatrix}$  which consists of both estimation error vector and state vector.

From (4.6) and (4.10), we deduce:

$$\begin{aligned} \dot{\zeta} = & \begin{bmatrix} (A - BK) & BK \\ \mathbb{O} & (A - LC) \end{bmatrix} \zeta + \begin{bmatrix} \mathbb{O} \\ \sum_{i,j=1}^{r,\bar{p}} g_{ij} (-LG) \mathcal{G}_{ij} G_i \end{bmatrix} \zeta \\ & + \begin{bmatrix} \sum_{i,j=1}^{m,\bar{n}} \bar{f}_{ij} F \mathcal{H}_{ij} H_i & \mathbb{O} \\ \mathbb{O} & \sum_{i,j=1}^{m,\bar{n}} \bar{f}_{ij} F \mathcal{H}_{ij} H_i \end{bmatrix} \zeta + \begin{bmatrix} E \\ E - LD \end{bmatrix} \omega. \end{aligned} \quad (4.11)$$

### Objective

The objective of this chapter is to determine the parameters  $L$  and  $K$  such that

- I) The close-loop system (4.11) is exponentially stable at  $t \rightarrow \infty$  when  $\omega = 0$ .
- II) When  $\omega \neq 0$ , the estimation error dynamic (4.11) converge asymptotically with maximum noise attenuation at  $t \rightarrow \infty$ .

A substantial amount of research has been performed to tackle this problem, resulting in the establishment of numerous methods in the literature, for example, [24, 95–97] and so on. Each of these methods has provided enhanced LMI conditions, which are formulated by deploying different mathematical tools. Regardless of these advancements in LMI relaxations, the resulting LMIs remain conservative; thus there is a scope for further enrichment. In this chapter, the authors have utilised the new-matrix multiplier-based LMI approach in the context of observer design, to establish a new and less conservative LMI condition compared to the existing results in the literature.

### 4.3 A robust circle-criterion based approach

This section is dedicated to the development of the LMI condition using the  $\mathcal{H}_\infty$  criterion. In order to mitigate the impact of external disturbances/noise  $\omega$  on the error dynamic, the  $\mathcal{H}_\infty$  criterion (3.17) is commonly used in the control system domain [12–14]. The  $\mathcal{H}_\infty$  criterion (3.17) for the estimation error dynamic (4.11) is modified as:

$$\|\zeta\|_{\mathcal{L}(2n)_2} \leq \sqrt{\mu\|\omega\|_{\mathcal{L}_2^q}^2 + \nu\|\zeta_0\|^2}. \quad (4.12)$$

In order to analyse the  $\mathcal{H}_\infty$  stability of the error dynamic (4.11), let us consider the ensuing quadratic Lyapunov function:

$$V(\tilde{x}) = \zeta^\top P \zeta, \quad (4.13)$$

where  $P = \begin{bmatrix} P_1 & \mathbb{O} \\ \mathbb{O} & P_2 \end{bmatrix} > 0$ , and  $P_1, P_2 \in \mathbf{S}^n$ .

It needs to be noted that more general Lyapunov matrices may be considered for the analysis, however, this will lead to bilinear matrix inequalities (BMIs), which are not suitable for this method. Thus, to avoid computation complexity and BMIs, the function defined in (4.13).

Further, the  $\dot{V}(\zeta(t))$  is computed along the trajectories of (4.11) and illustrated as follows:

$$\dot{V}(\zeta) = \zeta^\top \mathbf{A}_\zeta \zeta + \zeta^\top \begin{bmatrix} P_1 E \\ P_2 (E - LD) \end{bmatrix} \omega + \omega^\top \begin{bmatrix} P_1 E \\ P_2 (E - LD) \end{bmatrix}^\top \zeta, \quad (4.14)$$

where

$$\begin{aligned} \mathbf{A}_\zeta = & \begin{bmatrix} (A - BK)^\top P_1 + P_1 (A - BK) & P_1 BK \\ (BK)^\top P_1 & (A - LC)^\top P_2 + P_2 (A - LC) \end{bmatrix} \\ & + \begin{bmatrix} \sum_{i,j=1}^{m,\bar{n}} f_{ij} P_1 F \mathcal{H}_{ij} H_i & \mathbb{O} \\ \mathbb{O} & \sum_{i,j=1}^{m,\bar{n}} f_{ij} P_2 F \mathcal{H}_{ij} H_i \end{bmatrix} + \begin{bmatrix} \sum_{i,j=1}^{m,\bar{n}} (f_{ij} P_1 F \mathcal{H}_{ij} H_i) & \mathbb{O} \\ \mathbb{O} & \sum_{i,j=1}^{m,\bar{n}} (P_2 F \mathcal{F}_{ij} F_i) \end{bmatrix}^\top \\ & + \begin{bmatrix} \mathbb{O} \\ -\sum_{i,j=1}^{r,\bar{p}} g_{ij} P_2 L G \mathcal{G}_{ij} G_i \end{bmatrix} + \begin{bmatrix} \mathbb{O} \\ -\sum_{i,j=1}^{r,\bar{p}} g_{ij} P_2 L G \mathcal{G}_{ij} G_i \end{bmatrix}^\top. \end{aligned} \quad (4.15)$$

Similar to (3.20), the error dynamic (4.11) satisfies  $\mathcal{H}_\infty$  criterion (4.12) if the subsequent inequality is true:

$$\mathcal{W} \triangleq \dot{V}(\zeta) + \|\zeta\|^2 - \mu\|\omega\|^2 \leq 0. \quad (4.16)$$

One can refer to Remark 3.3 for more details about this.

Through the utilisation of (4.14) and (4.16), we obtain:

$$\mathcal{W} = \begin{bmatrix} \zeta \\ \omega \end{bmatrix}^\top \begin{bmatrix} \mathbf{A}_\zeta + \mathbb{I}_{2n} & \begin{bmatrix} P_1 E \\ P_2 (E - LD) \end{bmatrix} \\ \star & -\mu \mathbb{I}_q \end{bmatrix} \begin{bmatrix} \zeta \\ \omega \end{bmatrix} \leq 0. \quad (4.17)$$

Thus,  $\mathcal{W} \leq 0$  if

$$\begin{bmatrix} \mathbf{A}_\zeta + \mathbb{I}_{2n} & \begin{bmatrix} P_1 E \\ P_2 (E - LD) \end{bmatrix} \\ \star & -\mu \mathbb{I}_q \end{bmatrix} \leq 0. \quad (4.18)$$

In order to ensure the  $\mathcal{H}_\infty$  stability of (4.11), we need to satisfy the condition described in (4.18). Since it includes several terms obtained from the coupling between the decision variables  $P_1, P_2, L$ , and  $K$ , the inequality (4.18) is not easy to solve. Then, to overcome this problem, a convenient congruence transformation is utilised. By pre-multiplying and post-multiplying the

inequality (4.18) by  $\begin{bmatrix} \mathcal{P}_1 & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_n & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{I}_q \end{bmatrix}$  and its transpose, we deduce:

$$\begin{bmatrix} \bar{\mathbf{A}}_\zeta & \begin{bmatrix} E \\ P_2 E - R_L D \end{bmatrix} \\ \star & -\mu \mathbb{I}_q \end{bmatrix} \leq 0, \quad (4.19)$$

where

$$\begin{aligned} \bar{\mathbf{A}}_\zeta = & \begin{bmatrix} \mathcal{P}_1 A^\top + A \mathcal{P}_1 - R_K B^\top - B R_K^\top & \mathbb{O} \\ \mathbb{O} & A^\top P_2 + P_2 A - C^\top R_L - R_L^\top C \end{bmatrix} \\ & + \begin{bmatrix} \mathcal{P}_1 \mathcal{P}_1 & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_n \end{bmatrix} + \begin{bmatrix} \mathbb{O} & BK \\ (BK)^\top & \mathbb{O} \end{bmatrix} + \text{NL}_1 + \text{NL}_2, \end{aligned} \quad (4.20)$$

$$\text{NL}_1 = \sum_{i,j=1}^{m,\bar{n}} \underbrace{\begin{bmatrix} \mathcal{P}_1 H_i^\top & \mathbb{O} \\ \mathbb{O} & P_2 F \mathcal{H}_{ij} \end{bmatrix}}_{U_{ij}^\top} \underbrace{\begin{bmatrix} \bar{f}_{ij} \mathcal{H}_{ij}^\top F^\top & \mathbb{O} \\ \mathbb{O} & f_{ij} H_i \end{bmatrix}}_{V_{ij}} + V_{ij}^\top U_{ij}, \quad (4.21)$$

$$\text{NL}_2 = \sum_{i,j=1}^{r,\bar{p}} \left( \underbrace{\begin{bmatrix} \mathbb{O} \\ -R_L^\top G \mathcal{G}_{ij} \end{bmatrix}}_{M_{ij}^\top} \overbrace{\begin{bmatrix} g_{ij} G_i \end{bmatrix}}^{N_{ij}} + N_{ij}^\top M_{ij} \right), \quad (4.22)$$

and  $\mathcal{P}_1 = P_1^{-1}$ ,  $R_K^\top = K \mathcal{P}_1$  and  $R_L^\top = P_2 L$ .

Further, the term  $\begin{bmatrix} \mathcal{P}_1 \mathcal{P}_1 & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_n \end{bmatrix}$  and  $\begin{bmatrix} \mathbb{O} & BK \\ (BK)^\top & \mathbb{O} \end{bmatrix}$  are expressed in the following manner:

$$\begin{bmatrix} \mathcal{P}_1 \mathcal{P}_1 & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_n \end{bmatrix} = \underbrace{\begin{bmatrix} \mathcal{P}_1 & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_n \end{bmatrix}}_{\Sigma_P} \underbrace{\begin{bmatrix} \mathbb{I}_n & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_n \end{bmatrix}^{-1}}_{\Sigma_I} \begin{bmatrix} \mathcal{P}_1 & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_n \end{bmatrix}^\top, \quad (4.23)$$

and

$$\begin{bmatrix} \mathbb{O} & BK \\ (BK)^\top & \mathbb{O} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbb{O} \\ \mathbb{I} \end{bmatrix}}_{\mathbf{P}^\top} \overbrace{\begin{bmatrix} K^\top B^\top & \mathbb{O} \end{bmatrix}}^{\mathbf{Q}} + \mathbf{P}^\top \mathbf{Q}, \quad (4.24)$$

Now, the Young inequality (A.10) is exploited to linearize the terms  $\mathbb{K}$  and to tackle the non-linear terms  $\text{NL}_1$  and  $\text{NL}_2$ . First, the following notations are introduced to avoid cumbersome equations:

$$U = [U_{11} \ \cdots \ U_{1\bar{n}} \ \cdots \ U_{m1} \ \cdots \ U_{m\bar{n}}], \quad (4.25)$$

$$V = [V_{11} \ \cdots \ V_{1\bar{n}} \ \cdots \ V_{m1} \ \cdots \ V_{m\bar{n}}], \quad (4.26)$$

$$M = [M_{11} \ \cdots \ M_{1\bar{p}} \ \cdots \ M_{r1} \ \cdots \ M_{r\bar{p}}], \quad (4.27)$$

$$N = [N_{11} \ \cdots \ N_{1\bar{p}} \ \cdots \ N_{r1} \ \cdots \ N_{r\bar{p}}]. \quad (4.28)$$

Then, (4.21) and (4.22) are reformulated as:

$$\text{NL}_1 = U^\top V + V^\top U, \quad (4.29)$$

$$\text{NL}_2 = M^\top N + N^\top M. \quad (4.30)$$

The ensuing inequalities are deduced by employing (A.10) on (4.24), (4.29) and (4.30):

$$\mathbb{K} \leq (\mathbf{P} + \epsilon \mathcal{P}_1 \mathbf{Q})^\top (2\epsilon \mathcal{P}_1)^{-1} (\mathbf{P} + \epsilon \mathcal{P}_1 \mathbf{Q}), \quad (4.31)$$

$$\text{NL}_1 \leq (U + \bar{Z}V)^\top (2\bar{Z})^{-1} (U + \bar{Z}V), \quad (4.32)$$

$$\text{NL}_2 \leq (M + \mathbb{S}N)^\top (2\mathbb{S})^{-1} (M + \mathbb{S}N), \quad (4.33)$$

where  $\bar{Z} = \text{block-diag}(\mathbb{Z}, \bar{\mathbb{Z}})$ . The matrices  $\mathbb{Z}$ ,  $\bar{\mathbb{Z}}$  and  $\mathbb{S}$  are illustrated in (2.82), (4.34) and (2.83), respectively. Through utilisation of (3.13) (3.14) and (4.9), each elements inside  $V$  and  $M$  is

$$\bar{\mathbb{Z}} = \begin{bmatrix} \bar{Z}_{11} & \bar{Z}_{a_{12}} & \dots & \bar{Z}_{a_{1\bar{n}}} & \bar{Z}_{b_{21}^{11}} & \bar{Z}_{b_{22}^{11}} & \dots & \bar{Z}_{b_{2\bar{n}}^{11}} & \dots & \bar{Z}_{b_{m1}^{11}} & \bar{Z}_{b_{m2}^{11}} & \dots & \bar{Z}_{b_{m\bar{n}}^{11}} \\ \bar{Z}_{a_{12}} & \bar{Z}_{12} & \dots & \bar{Z}_{a_{1\bar{n}}^2} & \bar{Z}_{b_{21}^{12}} & \bar{Z}_{b_{22}^{12}} & \dots & \bar{Z}_{b_{2\bar{n}}^{12}} & \dots & \bar{Z}_{b_{m1}^{12}} & \bar{Z}_{b_{m2}^{12}} & \dots & \bar{Z}_{b_{m\bar{n}}^{12}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ \bar{Z}_{a_{1\bar{n}}^1} & \bar{Z}_{a_{1\bar{n}}^2} & \dots & \bar{Z}_{1\bar{n}} & \bar{Z}_{b_{21}^{1\bar{n}}} & \bar{Z}_{b_{22}^{1\bar{n}}} & \dots & \bar{Z}_{b_{2\bar{n}}^{1\bar{n}}} & \dots & \bar{Z}_{b_{m1}^{1\bar{n}}} & \bar{Z}_{b_{m2}^{1\bar{n}}} & \dots & \bar{Z}_{b_{m\bar{n}}^{1\bar{n}}} \\ \bar{Z}_{b_{21}^{11}} & \bar{Z}_{b_{21}^{12}} & \dots & \bar{Z}_{b_{21}^{1\bar{n}}} & \bar{Z}_{21} & \bar{Z}_{a_{22}^1} & \dots & \bar{Z}_{a_{2\bar{n}}^1} & \dots & \bar{Z}_{b_{m1}^{21}} & \bar{Z}_{b_{m2}^{21}} & \dots & \bar{Z}_{b_{m\bar{n}}^{21}} \\ \bar{Z}_{b_{22}^{11}} & \bar{Z}_{b_{22}^{12}} & \dots & \bar{Z}_{b_{22}^{1\bar{n}}} & \bar{Z}_{a_{22}^1} & \bar{Z}_{22} & \dots & \bar{Z}_{a_{2\bar{n}}^2} & \dots & \bar{Z}_{b_{m1}^{22}} & \bar{Z}_{b_{m2}^{22}} & \dots & \bar{Z}_{b_{m\bar{n}}^{22}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ \bar{Z}_{b_{2\bar{n}}^{11}} & \bar{Z}_{b_{2\bar{n}}^{12}} & \dots & \bar{Z}_{b_{2\bar{n}}^{1\bar{n}}} & \bar{Z}_{a_{2\bar{n}}^1} & \bar{Z}_{a_{2\bar{n}}^2} & \dots & \bar{Z}_{2\bar{n}} & \dots & \bar{Z}_{b_{m1}^{2\bar{n}}} & \bar{Z}_{b_{m2}^{2\bar{n}}} & \dots & \bar{Z}_{b_{m\bar{n}}^{2\bar{n}}} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \bar{Z}_{b_{m1}^{11}} & \bar{Z}_{b_{m1}^{12}} & \dots & \bar{Z}_{b_{m1}^{1\bar{n}}} & \bar{Z}_{b_{m1}^{21}} & \bar{Z}_{b_{m1}^{22}} & \dots & \bar{Z}_{b_{m1}^{2\bar{n}}} & \dots & \bar{Z}_{m1} & \bar{Z}_{a_{m2}^1} & \dots & \bar{Z}_{a_{m\bar{n}}^1} \\ \bar{Z}_{b_{m2}^{11}} & \bar{Z}_{b_{m2}^{12}} & \dots & \bar{Z}_{b_{m2}^{1\bar{n}}} & \bar{Z}_{b_{m2}^{21}} & \bar{Z}_{b_{m2}^{22}} & \dots & \bar{Z}_{b_{m2}^{2\bar{n}}} & \dots & \bar{Z}_{a_{m2}^1} & \bar{Z}_{m2} & \dots & \bar{Z}_{a_{m\bar{n}}^2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ \bar{Z}_{b_{m\bar{n}}^{11}} & \bar{Z}_{b_{m\bar{n}}^{12}} & \dots & \bar{Z}_{b_{m\bar{n}}^{1\bar{n}}} & \bar{Z}_{b_{m\bar{n}}^{21}} & \bar{Z}_{b_{m\bar{n}}^{22}} & \dots & \bar{Z}_{b_{m\bar{n}}^{2\bar{n}}} & \dots & \bar{Z}_{a_{m\bar{n}}^1} & \bar{Z}_{a_{m\bar{n}}^2} & \dots & \bar{Z}_{m\bar{n}} \end{bmatrix}, \quad (4.34)$$

where  $\bar{Z}_{ij} = \bar{Z}_{ij}^\top > 0 \in \mathbb{R}^{\bar{n} \times \bar{n}}$ ,  $\bar{Z}_{a_{ij}^k} = \bar{Z}_{a_{ij}^k}^\top \geq 0 \in \mathbb{R}^{\bar{n} \times \bar{n}} \forall i, k \in \{1, \dots, m\}, \&j \in \{1, \dots, \bar{n}\}$ ;  $\bar{Z}_{b_{ij}^{kj}} = \bar{Z}_{b_{ij}^{kj}}^\top \geq 0 \in \mathbb{R}^{\bar{n} \times \bar{n}}, \forall i \in \{2, \dots, m\}, k \in \{1, \dots, m-1\}, \&j \in \{1, \dots, \bar{n}\}$  such that  $\bar{\mathbb{Z}} > 0$ .

bounded and belongs to convex sets  $\mathcal{F}_m$  and  $\mathcal{G}_r$ , respectively. The set of vertices of  $\mathcal{F}_m$  and  $\mathcal{G}_r$  are given by

$$\mathcal{F}_{H_m} = \left\{ \{ \Phi_{11}, \dots, \Phi_{1\bar{n}}, \dots, \Phi_{m1}, \dots, \Phi_{m\bar{n}} \} : \Phi_{ij} \in [0, f_{b_{ij}}] \right\}, \quad (4.35)$$

$$\mathcal{F}_{G_r} = \left\{ \{ \Psi_{11}, \dots, \Psi_{1\bar{p}}, \dots, \Psi_{r1}, \dots, \Psi_{r\bar{p}} \} : \Psi_{ij} \in [0, g_{b_{ij}}] \right\}. \quad (4.36)$$

From convexity principal, the inequalities (4.32) and (4.33) are fulfilled if

$$\text{NL}_1 \leq \left[ (U + \bar{Z}V)^\top (2\bar{Z})^{-1} (U + \bar{Z}V) \right]_{V \in \mathcal{F}_{H_m}}, \quad (4.37)$$

$$\text{NL}_2 \leq \left[ (M + \mathbb{S}N)^\top (2\mathbb{S})^{-1} (M + \mathbb{S}N) \right]_{N \in \mathcal{F}_{G_r}}, \quad (4.38)$$

The deployment of (4.23), (4.31), (4.37) and (4.38) into (4.20) results in:

$$\begin{aligned} \bar{\mathbf{A}}_\zeta \leq & \begin{bmatrix} \mathcal{P}_1 A^\top + A \mathcal{P}_1 - R_K B^\top - B R_K^\top & \mathbb{O} \\ \mathbb{O} & A^\top P_2 + P_2 A - C^\top R_L - R_L^\top C \end{bmatrix} + \Sigma_P^\top \Sigma_I \Sigma_P \\ & + \begin{bmatrix} (\mathbf{P} + \epsilon \mathcal{P}_1 \mathbf{Q})^\top (2\epsilon \mathcal{P}_1)^{-1} (\mathbf{P} + \epsilon \mathcal{P}_1 \mathbf{Q}) \end{bmatrix} + \begin{bmatrix} (U + \bar{Z}V)^\top (2\bar{Z})^{-1} (U + \bar{Z}V) \end{bmatrix}_{V \in \mathcal{F}_{H_m}} \\ & + \begin{bmatrix} (M + \mathbb{S}N)^\top (2\mathbb{S})^{-1} (M + \mathbb{S}N) \end{bmatrix}_{N \in \mathcal{F}_{G_r}}. \end{aligned} \quad (4.39)$$

Further, one can obtain the following inequality by employing Schur Lemma 4 on (4.39):

$$\bar{\mathbf{A}}_\zeta \leq \begin{bmatrix} \Sigma_{11} & \Sigma_P^\top & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} \\ \star & -\Sigma_I & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \star & \star & -2\epsilon \mathcal{P}_1 & \mathbb{O} & \mathbb{O} \\ \star & \star & \star & -2\bar{Z} & \mathbb{O} \\ \star & \star & \star & \star & -2\mathbb{S} \end{bmatrix}_{V \in \mathcal{F}_{H_m}, N \in \mathcal{F}_{G_r}}, \quad (4.40)$$

where

$$\Sigma_{11} = \begin{bmatrix} \mathcal{P}_1 A^\top + A \mathcal{P}_1 - R_K B^\top - B R_K^\top & \mathbb{O} \\ \mathbb{O} & A^\top P_2 + P_2 A - C^\top R_L - R_L^\top C \end{bmatrix}, \quad (4.41)$$

$$\Sigma_{12} = (\mathbf{P} + \epsilon \mathcal{P}_1 \mathbf{Q}), \quad (4.42)$$

$$\Sigma_{13} = (U + \bar{Z}V), \quad (4.43)$$

$$\Sigma_{14} = (M + \mathbb{S}N). \quad (4.44)$$

Now, we are ready to state the following theorem:

#### Theorem 4.1

If the matrices  $\mathbb{Z}$ ,  $\bar{Z}$  and  $\mathbb{S}$  described in (2.82), (4.34) and (2.83), respectively, exist along with the matrices  $\mathcal{P}_1, P_2 > 0 \in \mathbf{S}^n$ , matrices  $R_K \in \mathbb{R}^{n \times s}$ ,  $R_L \in \mathbb{R}^{p \times n}$  and positive scalars  $\epsilon, \mu$  such that the following optimisation problem is solvable:

$$\begin{aligned} & \text{minimise } \mu \text{ subject to,} \\ & \begin{bmatrix} \Sigma_{11} & \Sigma_P & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} & \Sigma_q \\ \star & -\Sigma_I & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \star & \star & -2\epsilon \mathcal{P}_1 & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \star & \star & \star & -2\bar{Z} & \mathbb{O} & \mathbb{O} \\ \star & \star & \star & \star & -2\mathbb{S} & \mathbb{O} \\ \star & \star & \star & \star & \star & -\mu \mathbb{I}_q \end{bmatrix}_{V \in \mathcal{F}_{H_m}, N \in \mathcal{F}_{G_r}} \leq 0, \end{aligned} \quad (4.45)$$

where  $\Sigma_q = \begin{bmatrix} E^\top & (P_2 E - R_L^\top D)^\top \end{bmatrix}^\top$  and the terms  $\Sigma_{11}, \Sigma_P, \Sigma_{12}, \Sigma_{13}, \Sigma_{14}$  are specified in (4.41), (4.23), (4.42), (4.43) and (4.44), respectively. Then, the error dynamic (4.11) is  $\mathcal{H}_\infty$  asymptotically stable. The observer parameters are calculated as:  $K = R_K^\top \mathcal{P}_1^{-1}$  and  $L = P_2^{-1} R_L^\top$ .

**Proof:** One can derive the LMI (4.45) by substituting (4.40) in the inequality (4.19). According to the convexity principle [18], if the LMI (4.45) is solved for all  $V \in \mathcal{F}_{H_m}$  and  $N \in \mathcal{F}_{G_r}$ , then the error dynamic (4.11) satisfied the  $\mathcal{H}_\infty$  criterion (4.12) with  $\nu = \lambda_{\max}(P)$  and minimum  $\mu$  obtained from the solution of LMI (4.45). This ends the proof.  $\square$

The aforementioned theorem can be modified for the disturbance-affected nonlinear system with linear outputs, which is described in the subsequent Corollary.

#### Corollary 4.1

Let us consider the subsequent nonlinear systems having linear outputs:

$$\begin{aligned} \dot{x} &= Ax + Ff(x) + Bu + E\omega, \\ y &= Cx + D\omega, \end{aligned} \quad (4.46)$$

where all the variables and parameters remain consistent with those specified in (4.1). The ensuing observer-based controller is utilised for the stabilisation of the system (4.46):

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Ff(\hat{x}) + Bu + L(y - C\hat{x}), \\ u &= -K\hat{x}, \end{aligned} \quad (4.47)$$

if the following optimisation problem is solvable:

$$\begin{aligned} & \text{minimise } \mu \text{ subject to} \\ & \begin{bmatrix} \Sigma_{11} & \Sigma_P & \Sigma_{12} & \Sigma_{13} & \Sigma_q \\ \star & -\Sigma_I & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \star & \star & -2\epsilon\mathcal{P}_1 & \mathbb{O} & \mathbb{O} \\ \star & \star & \star & -2\bar{Z} & \mathbb{O} \\ \star & \star & \star & \star & -\mu\mathbb{I}_q \end{bmatrix}_{V \in \mathcal{F}_{H_m}} < 0, \end{aligned} \quad (4.48)$$

where all the variables are the same as the one specified in Theorem 4.1 .

For the proof, one can follow the proof of Theorem 4.1 .

## 4.4 ISS-based LMI formulation

In this segment, the novel LMI condition is derived by incorporating the ISS notion 19, which ensures the stability of the error dynamic (4.11).

In order to ensure the ISS behaviour of the closed-loop system (4.11), the quadratic Lyapunov function (4.13) is considered. It infers that  $V(\zeta)$  satisfies:

$$\lambda_{\min}(P)\|\zeta\|^2 \leq V(\zeta) \leq \lambda_{\max}(P)\|\zeta\|^2. \quad (4.49)$$

The derivative of  $V(\zeta(t))$  along the trajectories of (4.11) is showcased in (4.14). Further, the subsequent inequality is derived by deploying the Young inequality (A.9) on (4.14):

$$\dot{V}(\zeta) \leq \zeta^\top \left[ \mathbf{A}_\zeta + \frac{1}{\delta} \mathbb{Q}^\top \mathbb{Q} \right] \zeta + \left[ \begin{array}{c} P_1 E \\ P_2 (E - LD) \end{array} \right]^\top \zeta + \delta \omega^\top \omega, \quad (4.50)$$

where  $\mathbb{Q} = \begin{bmatrix} P_1 E \\ P_2 (E - LD) \end{bmatrix}^\top$ .

Now, let us presume that there exists a positive scalar  $\rho$  such that

$$\mathbf{A}_\zeta + \frac{1}{\delta} \mathbb{Q}^\top \mathbb{Q} + \rho P \leq 0. \quad (4.51)$$

Through the utilisation of Schur' lemma A.3, the condition specified in (4.51) is equivalent to

$$\begin{bmatrix} \mathbf{A}_\zeta & \mathbb{Q}^\top \\ \star & -\delta \mathbb{I}_q \end{bmatrix} \leq 0. \quad (4.52)$$

If the inequality (4.51) or (4.52) is true, then  $V(\zeta)$  fulfils :

$$\dot{V}(\zeta(t)) \leq -\rho V(\zeta(t)) + \delta \|\omega(t)\|^2 \leq -\rho \lambda_{\max}(P) \|\zeta(t)\|^2 + \delta \|\omega(t)\|^2. \quad (4.53)$$

The inequalities (4.49) and (4.53) are identical to the conditions described in (1.34) and (1.35), respectively. From Definition 20 of Chapter 1, the Lyapunov function (4.13) is an ISS-Lyapunov function along with the ensuing  $\mathcal{K}_\infty$  functions:

$$\begin{aligned} \alpha_1(\|\zeta(t)\|) &= \lambda_{\min}(P) \|\zeta(t)\|^2, \quad \alpha_2(\|\zeta(t)\|) = \lambda_{\max}(P) \|\zeta(t)\|^2, \\ \alpha_3(\|\zeta(t)\|) &= -\rho \lambda_{\max}(P) \|\zeta(t)\|^2 \text{ and } \alpha_4(\omega) = \delta \|\omega\|^2. \end{aligned}$$

By using Theorem 1.3 , the system (4.11) is an ISS w.r.t.  $\omega$  as it admits an ISS Lyapunov function (4.13).

One of the essential conditions required to ensure that the Lyapunov function (4.13) to be an ISS-Lyapunov function specified is provided in (4.52). Similar to the aforementioned section, the

inequality (4.52) is pre-multiplied and post-multiplied by the matrix  $\begin{bmatrix} \mathcal{P}_1 & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_n & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{I}_q \end{bmatrix}$  to eliminate

the coupling terms, and we get:

$$\begin{bmatrix} \bar{\Sigma} & \Sigma_q \\ \star & -\delta \mathbb{I}_q \end{bmatrix} \leq 0, \quad (4.54)$$

where

$$\begin{aligned} \bar{\Sigma} &= \begin{bmatrix} \mathcal{P}_1 A^\top + A \mathcal{P}_1 - R_K B^\top - B R_K^\top + \rho \mathcal{P}_1 & \mathbb{O} \\ \mathbb{O} & A^\top P_2 + P_2 A - C^\top R_L - R_L^\top C + \rho P_2 \end{bmatrix} \\ &+ \begin{bmatrix} \mathbb{O} & B K \\ (B K)^\top & \mathbb{O} \end{bmatrix} + \text{NL}_1 + \text{NL}_2, \end{aligned} \quad (4.55)$$

where  $\text{NL}_1$  and  $\text{NL}_2$  are specified in (4.21) and (4.22), respectively. Through the utilisation of (4.31), (4.37) and (4.38), the term  $\bar{\Sigma}$  is rewritten as:

$$\begin{aligned} \bar{\Sigma} &\leq \Sigma_1 + \begin{bmatrix} (\mathbf{P} + \epsilon \mathcal{P}_1 \mathbf{Q})^\top (2\epsilon \mathcal{P}_1)^{-1} (\mathbf{P} + \epsilon \mathcal{P}_1 \mathbf{Q}) \end{bmatrix} \\ &+ \begin{bmatrix} (U + \bar{Z}V)^\top (2\bar{Z})^{-1} (U + \bar{Z}V) \end{bmatrix}_{V \in \mathcal{F}_{H_m}} + \begin{bmatrix} (M + \mathcal{S}N)^\top (2\mathcal{S})^{-1} (M + \mathcal{S}N) \end{bmatrix}_{N \in \mathcal{F}_{G_r}}, \end{aligned} \quad (4.56)$$

where

$$\Sigma_1 = \begin{bmatrix} \mathcal{P}_1 A^\top + A \mathcal{P}_1 - R_K B^\top - B R_K^\top + \rho \mathcal{P}_1 & \mathbb{O} \\ \mathbb{O} & A^\top P_2 + P_2 A - C^\top R_L - R_L^\top C + \rho P_2 \end{bmatrix}. \quad (4.57)$$

Analogous to (4.40), the deployment of Schur lemma A.3 on (4.56) yields:

$$\bar{\Sigma} \leq \begin{bmatrix} \Sigma_1 & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} \\ \star & -2\epsilon\mathcal{P}_1 & \mathbb{O} & \mathbb{O} \\ \star & \star & -2\bar{\mathbb{Z}} & \mathbb{O} \\ \star & \star & \star & -2\mathbb{S} \end{bmatrix}_{V \in \mathcal{F}_{H_m}, N \in \mathcal{F}_{G_r}}, \quad (4.58)$$

where  $\Sigma_{13}, \Sigma_{14}$  are illustrated in (4.43) and (4.44), respectively. Now we can establish the following theorem illustrating an LMI that ensures the ISS behaviour of the error dynamic (4.11).

#### Theorem 4.2

Assume that there exist matrices  $\mathbb{Z}$ ,  $\bar{\mathbb{Z}}$ , and  $\mathbb{S}$  under the form (2.82), (4.34) and (2.83), respectively, along with  $\mathcal{P}_1, \mathcal{P}_2 > 0 \in \mathbf{S}^n$ , matrices  $R_K \in \mathbb{R}^{n \times s}$ ,  $R_L \in \mathbb{R}^{p \times n}$  and positive scalars  $\epsilon, \rho, \delta$  such that the following optimisation problem is solvable:

$$\begin{aligned} & \text{minimise } \mu \text{ subject to,} \\ & \begin{bmatrix} \Sigma_1 & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} & \Sigma_q \\ \star & -2\epsilon\mathcal{P}_1 & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \star & \star & -2\bar{\mathbb{Z}} & \mathbb{O} & \mathbb{O} \\ \star & \star & \star & -2\mathbb{S} & \mathbb{O} \\ \star & \star & \star & \star & -\delta\mathbb{I} \end{bmatrix}_{V \in \mathcal{F}_{H_m}, N \in \mathcal{F}_{G_r}} < 0, \end{aligned} \quad (4.59)$$

where  $\Sigma_q = \begin{bmatrix} E^\top & (P_2 E - R_L^\top D)^\top \end{bmatrix}^\top$  and the terms  $\Sigma_{11}, \Sigma_{12}, \Sigma_{13}, \Sigma_{14}$  are specified in (4.41), (4.42), (4.43) and (4.44), respectively. Then the system (4.11), with parameters of the observer-based controller computed as  $K = R_K^\top \mathcal{P}_1^{-1}$  and  $L = P_2^{-1} R_L^\top$ , is ISS with respect to  $\omega(t)$  according to the following inequality:

$$\|\zeta(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \|\zeta_0\| \zeta^{-\frac{\rho}{2}t} + \sqrt{\frac{\delta}{\rho \lambda_{\min}(P)}} \|\omega\|_{\mathcal{L}_2}. \quad (4.60)$$

**Proof:** LMI (4.59) is obtained by incorporating (4.58) in (4.52). From the convexity principle [18], if the LMI (4.45) is solved for all  $V \in \mathcal{F}_{H_m}$  and  $N \in \mathcal{F}_{G_r}$ , then inequality (4.52) is satisfied and it infers that the Lyapunov function (4.13) is an ISS-Lyapunov function. Through the integration of (4.53) and utilisation of comparison principle, one can get: we deduce that

$$\begin{aligned} V(\zeta(t)) & \leq V(\zeta_0) \zeta^{-\rho t} + \delta e^{-\rho t} \int_0^t e^{\rho s} \|\omega(s)\|_2^2 ds \\ & \leq V(\zeta_0) e^{-\rho t} + \delta \sup_{s \in [0, t]} \|\omega(s)\|_2^2 e^{-\rho t} \int_0^t e^{\rho s} ds \\ & \leq V(\zeta_0) e^{-\rho t} + \frac{\delta}{\rho} (1 - e^{-\rho t}) \sup_{s \in [0, t]} \|\omega(s)\|_2^2. \end{aligned} \quad (4.61)$$

Since  $0 \leq 1 - e^{-\rho t} \leq 1$ ,  $\sup_{s \in [0, t]} \|\omega(s)\|_2^2 \leq \|\omega\|_{\mathcal{L}_2}^2$  and  $\lambda_{\min}(P) \|\zeta\|^2 \leq V(\zeta) \leq \lambda_{\max}(P) \|\zeta\|^2$ , the inequality (4.60) is deduced. Hence, Theorem 4.2 is proved.  $\square$

Analogous to the aforementioned section, the proposed LMI (4.59) can be modified for the system (4.46), which is showcased in the subsequent corollary.

#### Corollary 4.2

The proposed observer-based controller (4.47) is deployed for the stabilisation of the system (4.46) if the following optimisation problem is solvable:

$$\begin{aligned} & \text{minimise } \delta \text{ subject to} \\ & \begin{bmatrix} \Sigma_1 & \Sigma_{12} & \Sigma_{13} & \Sigma_q \\ \star & -2\epsilon\mathcal{P}_1 & \mathbb{O} & \mathbb{O} \\ \star & \star & -2\bar{Z} & \mathbb{O} \\ \star & \star & \star & -\mu\mathbb{I}_q \end{bmatrix}_{V \in \mathcal{F}_{H_m}} < 0, \end{aligned} \quad (4.62)$$

where all the variables remain coherent with the one specified in Theorem 4.2 .

In the sequel, the effectiveness of the proposed observer-based controller is analysed through numerical examples.

## 4.5 Validating the performance of the developed methodology

This section focuses on emphasising the efficiency of the proposed methodology through two numerical examples.

### 4.5.1 Example 1: A single-link flexible joint robot model

The authors of [19, 98] have illustrated the model of a single-link flexible joint robot in the subsequent manner:

$$\begin{aligned} \dot{\theta}_m &= \omega_m, \\ \dot{\omega}_m &= \frac{k}{J_m}(\theta_l - \theta_m) - \frac{B}{J_m}\omega_m + \frac{K_\tau}{J_m}u, \\ \dot{\theta}_l &= \omega_l, \\ \dot{\omega}_l &= -\frac{K}{J_l}(\theta_l - \theta_m) - \frac{mgh}{J_l}\sin(\theta_l), \end{aligned} \quad (4.63)$$

where  $\theta_l$  and  $\theta_m$  depict the angular rotations of the link and motor of the robot arm, respectively. Whereas,  $\omega_l$  and  $\omega_m$  represent the angular velocities of the link and motor, respectively. The remaining parameters and their values are showcased in Table 4.1.

Further, the system specified in (4.63) can be written in the form of (4.46) with the ensuing the

$$\text{parameters: } x = \begin{bmatrix} \theta_m \\ \omega_m \\ \theta_l \\ \omega_l \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{bmatrix}, F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix}, E = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } f(x) = -15.86 \cdot m \cdot \sin(x_3).$$

Let us assume  $\rho = 1.35$  and  $\epsilon = 0.1$  for solving the LMIs (4.62). Through the utilisation of YALMIP toolbox [99], the feasibility of the LMIs (4.62) and [24, LMI (22)] is tested for different

Table 4.1: Parameters of the model of a single-link flexible joint robot

No.	Parameters	Value	Unit
1	$m$ - Mass of the pointer	0.21	kg
2	$2h$ - Length of the link	$3 \times 10^{-2}$	m
3	$g$ - Acceleration due to gravity	9.8	$\text{m} \cdot \text{sec}^{-2}$
4	$k$ - Torsional spring constant	$1.8 \times 10^{-1}$	$\text{N} \cdot \text{m} \cdot \text{rad}^{-1}$
5	$K_\tau$ - Amplifier gain	$8 \times 10^{-2}$	$\text{N} \cdot \text{m} \cdot \text{V}^{-1}$
6	$B$ - Viscous friction coefficient	$4.6 \times 10^{-2}$	$\text{N} \cdot \text{m} \cdot \text{V}^{-1}$
7	$J_l$ - Inertia of the Link	$9.3 \times 10^{-3}$	$\text{kg} \cdot \text{m}^2$
8	$J_m$ - Motor inertia	$3.7 \times 10^{-3}$	$\text{kg} \cdot \text{m}^2$

values of  $m$ . The obtained results are displayed in Table 4.2. It infers the developed LMI provides the feasible solution for larger values of  $m$ , whereas the one presented in [24] for the same values of  $m$  is infeasible. Hence, the developed LMI can provide a feasible solution for a larger Lipschitz constant as compared to the existing methods which are infeasible for the same constant. Thus, the efficiency of the proposed matrix-multiplier-based LMIs is highlighted.

Table 4.2: Comparison results for different values of  $m$ .

LMI approaches	Parameters	$m = 0.21$	$m = 0.3$	$m = 0.7$	$m = 1$
LMI (4.62)	$\delta$	9.7819 $\times 10^{-9}$	1.2058 $\times 10^{-8}$	5.7400 $\times 10^{-7}$	5.3920 $\times 10^{-5}$
	$\sqrt{\frac{\delta}{\rho \lambda_{\min}(P)}}$	2.5564	2.7852	1.839 $\times 10^1$	1.552 $\times 10^2$
[24, LMI (22)]	$\sqrt{\mu}$	2.5745	3.2291	Inf	Inf

Further, the proposed observer-based controller (4.47) is deployed to stabilise the states of the system in the MATLAB environment for the case  $m = 0.21$ . The solution of the LMI (4.62) yields the ensuing observer-based controller parameters:

$$L = \begin{bmatrix} 20.5758 & -19.5758 \\ -2.4059 & 3.4059 \\ 9.4141 & -8.4141 \\ 7.4211 & -6.4211 \end{bmatrix} \text{ and } K = [3.5343 \quad 0.8313 \quad -2.3666 \quad 0.7819].$$

The initial condition is given as  $x_0 = [1 \quad 1 \quad 1 \quad 1]^\top$ . Assume that the dynamics and outputs of the system are affected by the Gaussian noise  $\omega \rightsquigarrow (0, 0.01)$ . The behaviour of the estimation errors is shown in Figure 4.2, which highlights the asymptotic convergence of the estimation errors with optimal noise attenuation. Further, Figure 4.3 depicts the plot of stabilised states obtained from the observer-based controller and it emphasises the stabilisation of states. Thus, the performance of the observer-based controller is validated.

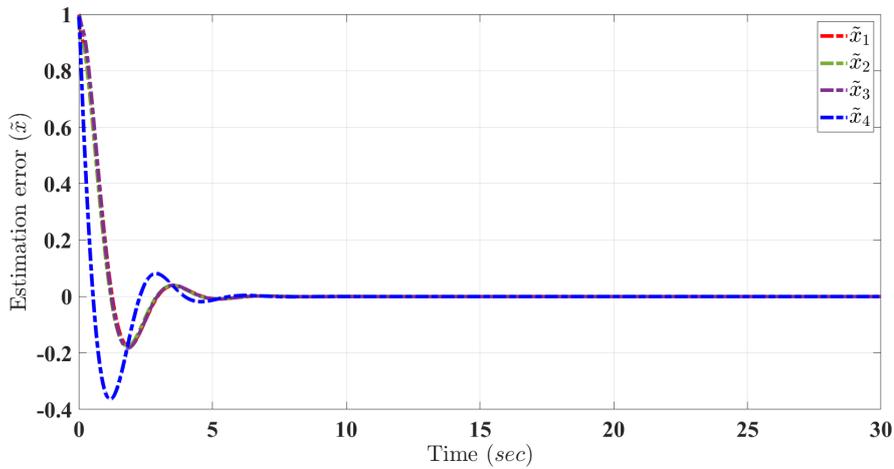


Figure 4.2: Graphical representation of Estimation error ( $\tilde{x}$ ) in case of the model of a single-link joint robot

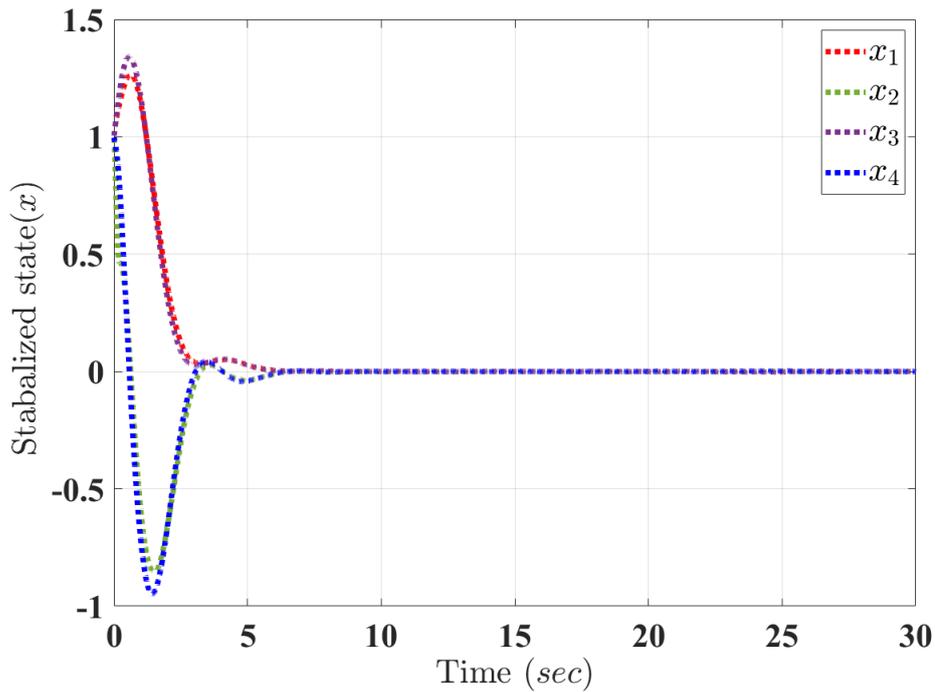


Figure 4.3: Graphical presentation of stabilised states ( $\tilde{x}$ ) for the application of a single-link robot

#### 4.5.2 Example 2: A numerical example of unstable system

In this segment of the chapter, a numerical example is utilised for analysing the performance of the proposed observer-based stabilisation approach. Let us consider the system (4.46) with the

subsequent parameters:

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, G = \begin{bmatrix} 1 & -0.1 \\ 0 & 0 \end{bmatrix}, E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix} \text{ and } D = 0.1.$$

The nonlinear function  $f(x)$  is illustrated as:  $f(x) = \begin{bmatrix} \sin(H_1 x) \\ \sin(H_2 x) \end{bmatrix}$ , where  $H_1 = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$  and  $H_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ . The initial value of the state vector is  $x_0 = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$ . Let us assume that the dynamics and outputs of the systems are corrupted with the Gaussian noise ( $\omega \rightsquigarrow (0, 0.01)$ ). For

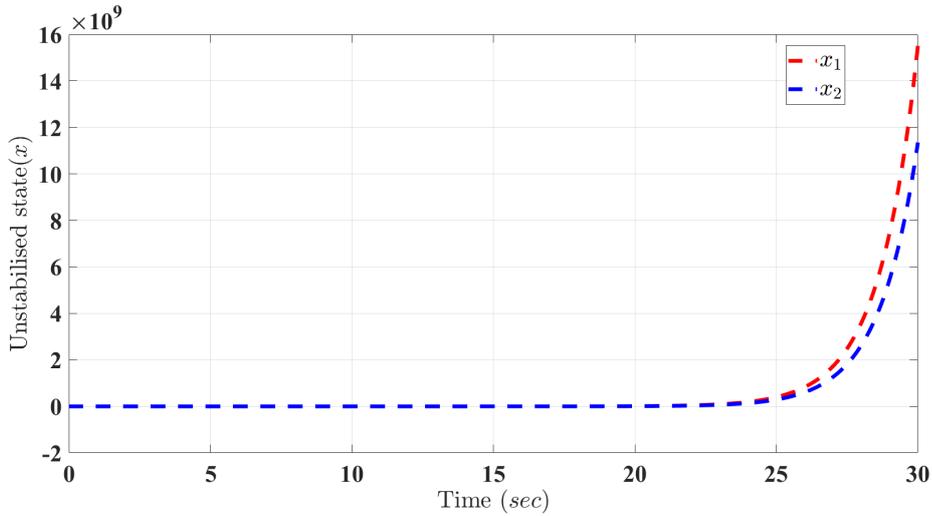


Figure 4.4: Visualisation of unstable states

input  $u = 1$ , the plot of states of the given system is displayed in Figure 4.4 and it infers that the states of the systems are not stable. Thus, to stabilise this system, the proposed methodology is deployed.

Table 4.3: Recapitulating solutions obtained from different LMIs

LMI (4.62)		LMI (4.48)	[24, LMI (22)]
$\delta$	$\gamma = \sqrt{\frac{\delta}{\rho \lambda_{\min}(P)}}$	$\sqrt{\mu}$	$\sqrt{\mu}$
$1.6186 \times 10^{-9}$	0.2804	0.1844	0.3540

The established LMIs (4.48), (4.62) and [24, LMI (22)] are solved using YALMIP toolbox of MATLAB [99] and the achieved results are outlined in Table 4.3. It showcases that the proposed LMIs provide a better value of noise attenuation level than the method of [24]. Thus, it highlights the superiority of the developed matrix-multiplier-based LMI approach over the existing methods.

In addition to this, the following parameters are computed from the solution of LMI (4.48):

$$L = \begin{bmatrix} 13.7839 \\ 13.1622 \end{bmatrix} \text{ and } K = \begin{bmatrix} 10.5423 & 7.4572 \end{bmatrix}.$$

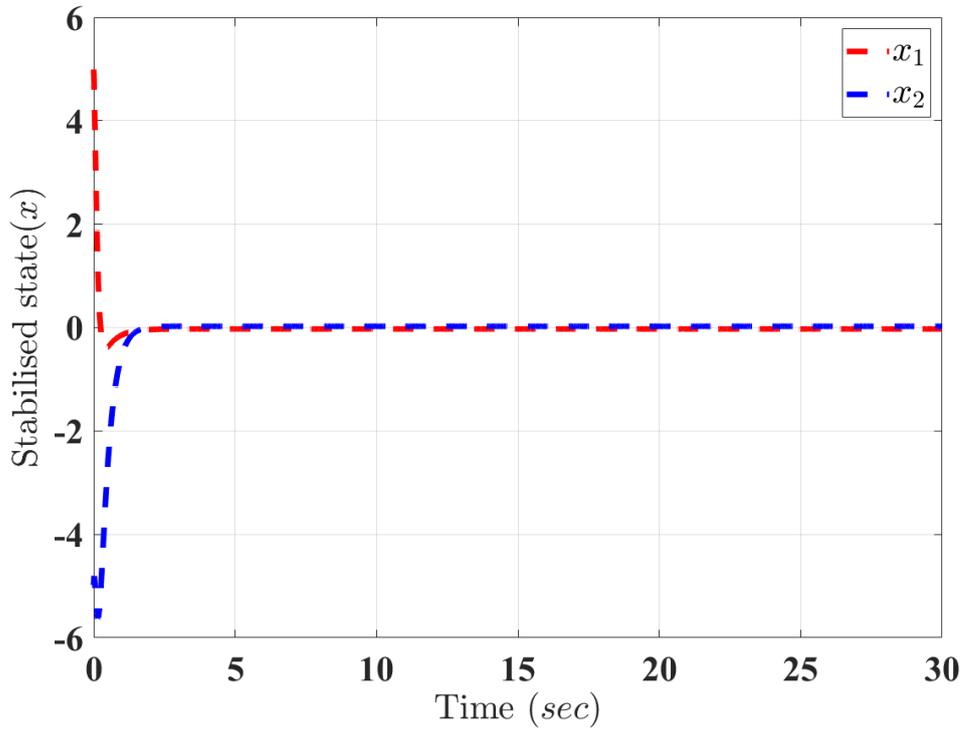


Figure 4.5: Representation of stabilised states

Further, for stabilisation of this system, the proposed observer-based controller (4.47) is de-

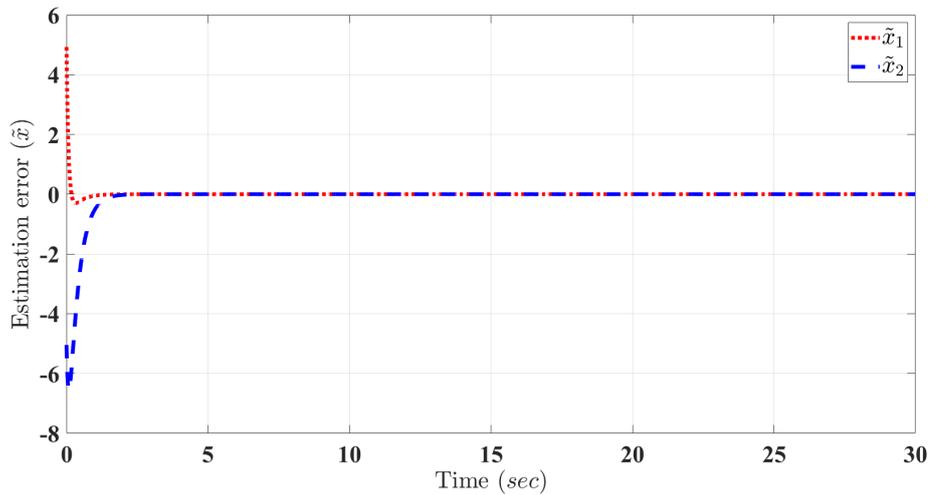


Figure 4.6: Plot of estimation error ( $\tilde{x}$ )

ployed in a MATLAB environment by using the aforementioned gain matrices. The obtained results are shown in Figure 4.5 and Figure 4.6. The behaviour of the estimation error is illustrated in Figure 4.5, and it highlights the asymptotic convergence of the estimation error with optimal noise attenuation. Figure 4.6 depicts the graphs of stabilised states and it emphasises

---

the rapid stabilisation of the states. Hence, the performance of the proposed observer-based controller is verified.

## 4.6 Conclusion

In this chapter, two novel LMI-based methods for designing the observer-based stabilization strategy are established. The first LMI condition is developed by deploying the  $\mathcal{H}_\infty$  criterion, whereas the second one is derived by using ISS notions. The obtained LMI conditions are less conservative than those existing in the literature. This is due to the prudent utilization of well-known mathematical tools such as a reformulated Lipschitz property, the Young relation, and then new matrix-multiplier techniques in a judicious way. The effectiveness of the proposed control strategy over the existing LMI-based methodology is shown through numerical examples. It implies that the feasibility of the proposed LMIs is enhanced.



---

# Nonlinear observer design for autonomous vehicle models: exploiting the proposed LMI approaches

## Contents

---

<b>5.1</b>	<b>Introduction</b>	<b>111</b>
<b>5.2</b>	<b>state estimation of the longitudinal autonomous vehicle model</b>	<b>112</b>
5.2.1	Longitudinal vehicle dynamics	112
5.2.2	Observer-based state estimation	114
<b>5.3</b>	<b>Slip angle estimation using ISS-based observer</b>	<b>117</b>
5.3.1	Lateral dynamics of vehicle	117
5.3.2	Employing ISS-based observer	118
<b>5.4</b>	<b>Transformation-based observer for vehicle tracking model</b>	<b>120</b>
5.4.1	Vehicle model	121
5.4.2	Observer development	121
<b>5.5</b>	<b>Conclusion</b>	<b>125</b>

---

## 5.1 Introduction

The development of communication and robotic technology has substantially impacted our day-to-day lifestyles, especially in the transportation sector. These technological advancements pioneered a path for the rise of autonomous vehicles (AVs). The primary objectives of these vehicles are listed as follows:

1. to mitigate the number of crashes, or to improve the safety of the driver;
2. to minimize energy consumption and pollution;
3. avoid traffic congestion.

The last few decades have witnessed growth in autonomous vehicle technologies which can be seen in [100–102]. In addition to this, according to [103], Google launched a project of Google self-driving car in 2009. In 2020, Uber initiated testing of self-driving vehicles through a partnership with Volvo [104]. All these cited papers prove the potential for further enhancements in the sector of smart vehicles. It motivates the authors to implement the aforementioned developed methodologies in the application of the autonomous vehicle.

The task of control of autonomous vehicles is mainly divided into two parts:

- I) **Longitudinal control**: It is primarily focused on control of the longitudinal motion of the vehicle, such as, its velocity or position w.r.t. to preceding vehicle in platoon, and so on. It plays a pivotal role in the control of connected autonomous vehicle platforms.
- II) **Lateral control**: It is mainly developed for avoiding lane departure accidents. There are three types of tasks performed by these systems such as lane departure warning, lane keeping and yaw stability control.

For each task, there has been a significant amount of research carried out in the literature [105–107]. For all these tasks, there are several approaches proposed in the literature. As there have been continuous advancements in the autonomous vehicle industry, it encourages authors to deploy the established methods for the numerous tasks in autonomous vehicles. In this chapter, the authors are mainly focused on the implementation of the proposed observer methodology for several tasks in autonomous vehicles, for example, longitudinal state estimations and slip angle estimations, etc. It aids in highlighting the effectiveness of the developed techniques in the real-time application domain (autonomous vehicle industry).

The remaining part of this chapter is structured as follows: Section 5.2 is dedicated to the implementation of the proposed observers of Chapter 3 on the longitudinal vehicle model. The deployment of the ISS-based observer of Chapter 3 for the purpose of slip-angle estimation of the nonlinear vehicle model is illustrated in Section 5.3. Further, Section 5.4 encompasses the development of the coordinate transformation-based nonlinear observer for a vehicle tracking model. Finally, some concluding remarks are provided in Section 5.5.

## 5.2 state estimation of the longitudinal autonomous vehicle model

The control of autonomous vehicles is an important and popular topic ([108–110]). The vehicle control system requires the knowledge of the longitudinal vehicle states, i.e., velocity, position, and acceleration, to control various vehicle parameters/states such as consecutive distances between two vehicles, vehicle yaw rates, and side slip angles. Thus, in order to achieve efficient vehicle control, the longitudinal vehicle states must be estimated. Several autonomous vehicle models have been established in the literature for vehicle control tasks, which are presented in [109, 111–113]. Table 5.1 describes a few models that portray the longitudinal motion of vehicles.

### 5.2.1 Longitudinal vehicle dynamics

The commonly used vehicle dynamics for longitudinal control are listed as follows:

Table 5.1: The models to depict the vehicular longitudinal motion

No.	Model	References	Features	Drawbacks
1	"Position-Velocity" second-order linear model	[112], [114]	Simple in the form as compared to other models	Lacks power-train dynamics; less accurate
2	"Position-Velocity" second-order nonlinear model	[113], [115]		
3	"Position-Velocity-Force" third-order nonlinear model	[109], [116]	consist single nonlinear term which reduces the complexity	difficult to measure force term through sensors
4	"Position-Velocity-Acceleration" third-order nonlinear model	[111], [117]	simple nonlinear terms; high accuracy	
5	"Position-Velocity-Acceleration" third-order linear model	[118], [119]	easy to implement in practise	needs an exact information about vehicles

1. **Position-Velocity-Force** third-order nonlinear model ( [108], [109]):

$$\begin{aligned}
\dot{p}(t) &= v(t), \\
\dot{v}(t) &= \frac{1}{m}F(t) - \frac{K_d}{m}v^2(t) - \frac{d_m}{m}, \\
\dot{F}(t) &= \frac{1}{\tau}u(t) - \frac{1}{\tau}F(t),
\end{aligned} \tag{5.1}$$

where  $p(t)$ ,  $v(t)$  and  $F(t)$  indicate the position, velocity and driving/braking force of the vehicle, respectively.  $u(t)$  denotes the control input (desired driving/braking force or desired acceleration). Other parameters of the model are specified in Table 5.2.

2. **Position-Velocity-Acceleration** third-order nonlinear model ( [120], [121]):

$$\begin{aligned}
\dot{p}(t) &= v(t), \\
\dot{v}(t) &= a(t), \\
\dot{a}(t) &= -\frac{1}{\tau}a(t) - \frac{2K_d}{\tau m}v^2(t) - \frac{2K_d}{m}v(t)a(t) + \frac{1}{m\tau}u(t) - \frac{d_m}{\tau m},
\end{aligned} \tag{5.2}$$

where  $a(t)$  implies the acceleration of the vehicle. Other variables and parameters remain consistent with the one specified in (5.1).

3. **Position-Velocity-Acceleration** third-order linear model ( [119], [122]):

$$\begin{aligned}
\dot{p}(t) &= v(t), \\
\dot{v}(t) &= a(t), \\
\dot{a}(t) &= -\frac{1}{\tau}a(t) - \frac{2K_d}{\tau m}v^2(t) - \frac{2K_d}{m}v(t)a(t) + \frac{1}{m\tau}u(t) - \frac{d_m}{\tau m},
\end{aligned} \tag{5.3}$$

where all variables and parameters are the same as the one described in (5.2).

Table 5.2: Vehicle parameters

No.	Parameter of vehicle	Range	Unit
1	$m$ - Mass	950 - 1750	kg
2	$\tau$ - Engine time lag	0.2 - 0.6	sec
3	$K_d$ - Aerodynamics drag coefficient	0.1-0.5	$\frac{\text{Nsec}^2}{\text{m}^2}$
4	$d_m$ - Mechanical drag	0 - 110	N

Among these models, the system (5.3) is frequently used in the longitudinal state estimation problem. However, it is deduced from the exact feedback linearization of the nonlinear vehicle plant (5.2) under the assumption that all the vehicular parameters are known. It is difficult to obtain complete information on vehicle dynamics in practical scenarios. Thus, the model (5.2) is considered over dynamics (5.2) and (5.1) for the longitudinal control of vehicles. In addition to this, the acceleration term utilised in (5.2) is easy to measure as compared to the  $F(t)$  of (5.1), and the nonlinear system (5.2) is easy to deploy than.

The system output is given by

$$\bar{y} = p.$$

Let us consider that both dynamics and measurements of the model are assumed to be disturbance-corrupted by  $\omega$ .

Therefore, the state-space representation of (5.2) is:

$$\begin{aligned}\dot{\eta} &= A_\eta \eta + G_\eta f(\eta) + B_\eta u + E_\eta \omega, \\ \bar{y} &= C_\eta \eta + D_\eta \omega,\end{aligned}\tag{5.4}$$

$$\text{where } \eta = \begin{bmatrix} p \\ v \\ a \end{bmatrix}, f(\eta) = \begin{bmatrix} v^2(t) \\ v(t)a(t) \end{bmatrix}, E_\eta = \begin{bmatrix} 0 \\ 0 \\ 0.25 \end{bmatrix}, B_\eta = \begin{bmatrix} 0 \\ 0 \\ \frac{-1}{m\tau} \end{bmatrix}, A_\eta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \frac{-1}{\tau} \end{bmatrix},$$

$$G_\eta = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -\frac{2K_d}{\tau m} & -\frac{2K_d}{m} \end{bmatrix}, C_\eta = [1 \ 0 \ 0] \text{ and } D_\eta = [-1].$$

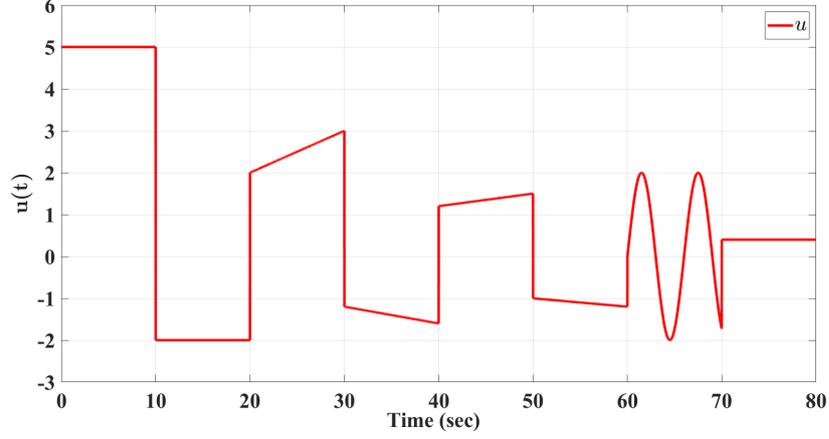
### 5.2.2 Observer-based state estimation

The vehicular plant expressed in (5.2) is already in the form of (3.55) and (3.123). The vehicle parameters are as follows:

$$m = 1000 \text{ kg}, \tau = 0.5 \text{ sec}, K_d = 0.25 \frac{\text{Nsec}^2}{\text{m}^2}, \text{ and } d_m = 100 \text{ N}.$$

The input  $u(t)$  of the vehicle dynamics (5.2) is showcased in Figure 5.1. For the given input and initial condition  $x_0 = [p_0 \ v_0 \ a_0]^\top = [25 \ 2 \ 2]^\top$ , the states of the vehicle are bounded in the subsequent set:

$$\begin{aligned}\mathcal{V}_n &= \left\{ x : x \in \{ -53.9192 \leq x_1(=p) \leq 71.0380, \right. \\ &\quad \left. -5.1548 \leq x_2(=v) \leq 2.8490, \right. \\ &\quad \left. \text{and } -0.1127 \leq x_3(=a) \leq 1.9790 \} \right\}, \forall t \in [0, 80].\end{aligned}$$

Figure 5.1: Input  $u(t)$  given to nonlinear vehicle model

$\forall x \in \mathcal{V}_n$ , the partial derivatives of nonlinearities ( $f_1 = x_2^2$  and  $f_2 = x_2 x_3$ ) hold:

$$-10.309 \leq \frac{\partial f_1}{\partial x_2} \leq 5.698, \quad -5.154 \leq \frac{\partial f_2}{\partial x_3} \leq 2.849, \quad \text{and} \quad -0.1127 \leq \frac{\partial f_2}{\partial x_2} \leq 1.979.$$

Consider matrices  $H_1$  and  $H_2$  of the nonlinearities  $f_1(H_1 x)$  and  $f_2(H_2 x)$  as:

$$H_1 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad H_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus,  $m = 2$  and  $\bar{n} = 2$ .

Let the system dynamics and outputs be affected by the Gaussian noise  $\omega \rightsquigarrow (0, 0.1)$  for  $t \in [0, 80]$ .

Let us consider the following cases for the analysis of the proposed LMIs:

1. **Case 1:** LMI (3.58) along with

$$\mathbb{Z} = \begin{bmatrix} Z_{11} & \alpha Z_{21} & \alpha Z_{22} \\ \star & Z_{21} & \alpha Z_{22} \\ \star & \star & Z_{22} \end{bmatrix}, \quad (5.5)$$

where  $\alpha = 0.1$  and  $Z_{11}, Z_{21}, Z_{22} \in \mathbb{R}^{\bar{n} \times \bar{n}}$  are symmetric positive definite matrices so that  $\mathbb{Z} > 0$ .

2. **Case 2:** LMI (3.58) with the proposed matrix multiplier,

$$\mathbb{Z} = \begin{bmatrix} Z_{11} & Z_{b_{21}^{11}} & Z_{b_{22}^{11}} \\ \star & Z_{21} & Z_{a_{22}^1} \\ \star & \star & Z_{22} \end{bmatrix}, \quad (5.6)$$

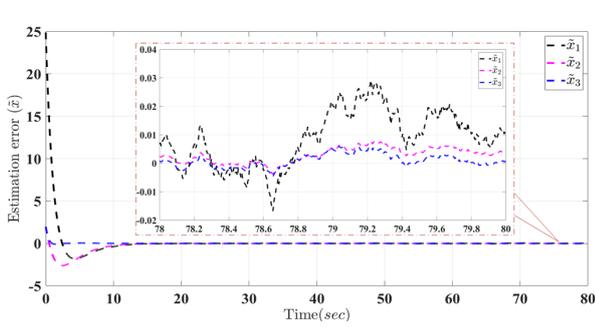
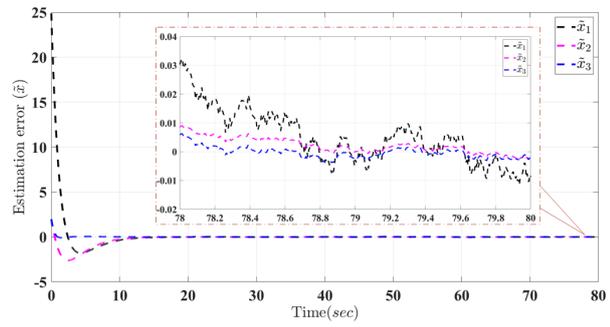
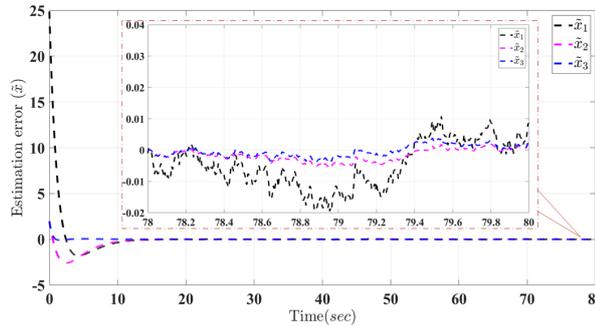
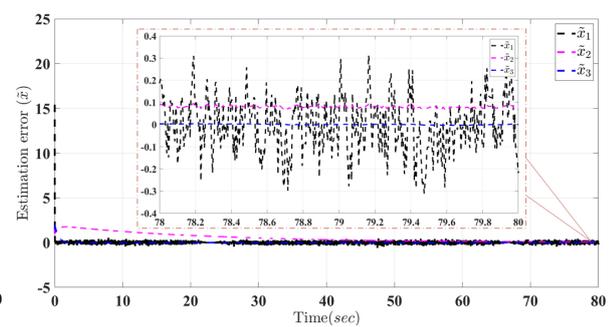
where  $Z_{11}, Z_{21}, Z_{22}, Z_{b_{21}^{11}}, Z_{b_{22}^{11}}, Z_{a_{22}^1} \in \mathbb{R}^{\bar{n} \times \bar{n}}$  such that  $\mathbb{Z} > 0$ .

3. **Case 3:** LMI (3.59) with the matrix expressed in (5.6).

4. **Case 4:** LMI approach developed in [73].

Table 5.3: Obtained optimal values in different cases

	Case 1	Case 2	Case 3	Case 4
$\sqrt{\mu}$	1.4323	1.4312	1.4276	7.5876
$L$	$\begin{bmatrix} 1.1006 \\ 0.2568 \\ 0.0516 \end{bmatrix}$	$\begin{bmatrix} 1.1010 \\ 0.2567 \\ 0.0515 \end{bmatrix}$	$\begin{bmatrix} 1.1008 \\ 0.2556 \\ 0.0515 \end{bmatrix}$	$\begin{bmatrix} 99.0453 \\ 4.2119 \\ 0.1697 \end{bmatrix}$

(a) Plot of  $\tilde{x}$  in Case 1(b) Plot of  $\tilde{x}$  in Case 2(c) Plot of  $\tilde{x}$  in Case 3(d) Plot of  $\tilde{x}$  in Case 4Figure 5.2: Trajectories of estimation error ( $\tilde{x}$ ) in several cases

Through the utilisation of the MATLAB toolbox, the gain matrix in each case is computed. The solutions obtained in all the above cases are outlined in Table 5.3. It showcases that the optimal value of  $\sqrt{\mu}$  obtained in Case 3 is minimum as compared to other cases. It infers that the noise attenuation achieved by the proposed LMI (3.59) is better than the other existing approaches. Further, in the MATLAB environment, the observer specified in (3.56) is deployed on the aforementioned vehicle model for the purpose of longitudinal state estimation. The graphical representation of the estimation error ( $\tilde{x}$ ) of the developed observer in the above cases is depicted in Figure 5.2. The noise compensation displayed in Figures 5.2c, 5.2b, 5.2a is better as compared to the same in Figure 5.2d. It implies that one can efficiently estimate the longitudinal states of autonomous vehicles with the developed matrix-multipliers-based LMIs with optimal noise compensation. Therefore, the performance of the proposed LMI-based observer is validated.

### 5.3 Slip angle estimation using ISS-based observer

In the automobile domain, researchers developed several vehicle control systems to inhibit vehicles from drifting out, rolling over and spinning. It is mainly classified as:

1. **Yaw stability control system:** It avoid skidding and spinning out of the vehicle.
2. **Roll stability control system:** It prevents roll-over of an vehicle.

In addition to this, there are numerous integrated control systems which perform both earlier-mentioned tasks. According to [114], on low-friction road surfaces, it is beneficial to control vehicle slip angle rather than yaw angle for lateral stability of vehicles. The feedback control of vehicle slip angle is essential as its excessive value can hamper the tire's ability to produce lateral forces, potentially exposing the vehicle to danger. Consequently, both yaw rate and vehicle slip angle are vital components in the control of autonomous vehicles. Due to the high price of sensors, slip angles can not be measured directly in the case of autonomous vehicles. Thus, model-based estimation is performed to determine the slip angle. The primary objective of this section is to deploy the proposed observer (3.71) for the slip angle estimation.

#### 5.3.1 Lateral dynamics of vehicle

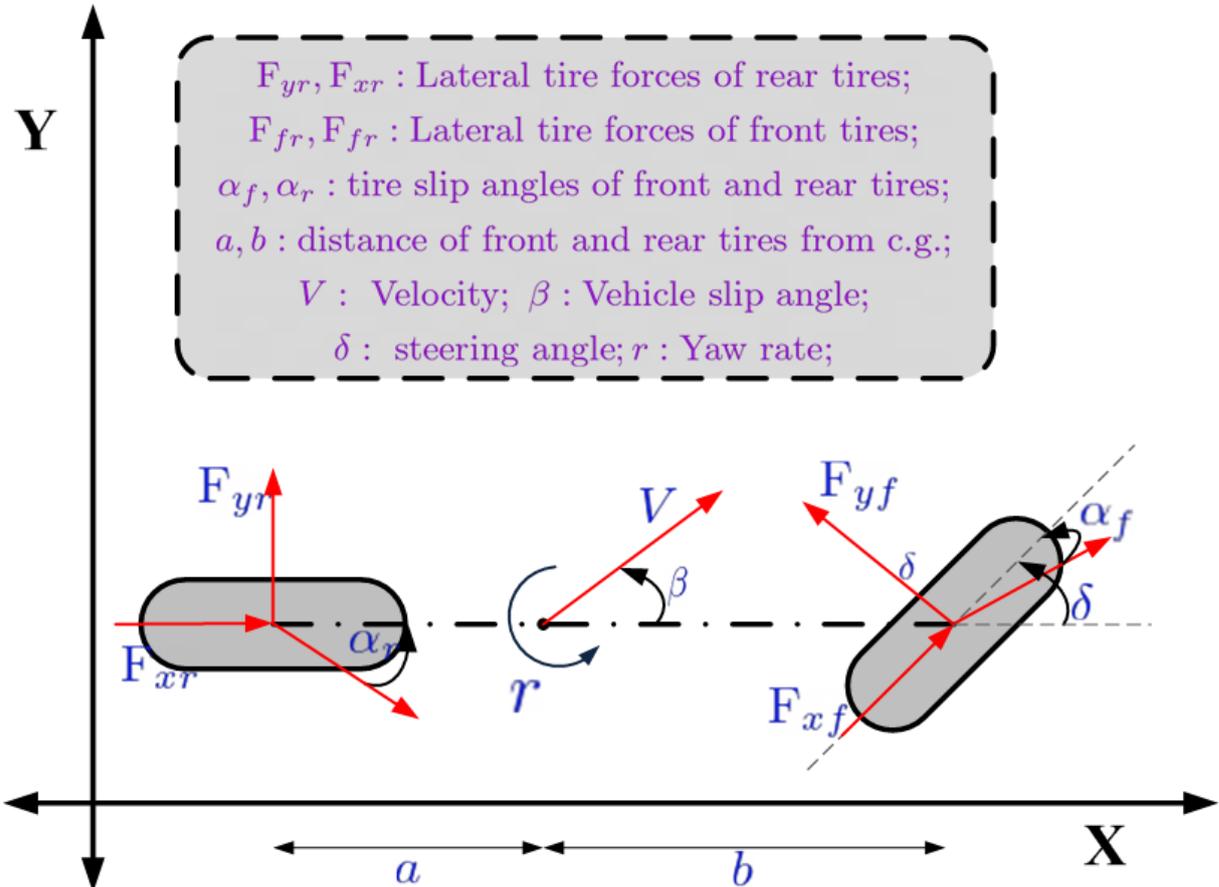


Figure 5.3: Pictorial representation of a single track model for lateral vehicle dynamic

The lateral vehicle dynamic with 2 DOF is showcased in Figure 5.3. It includes lateral translation and rotational yaw motion of the vehicle. The ensuing set of equations represents the nonlinear lateral vehicle dynamics:

$$\begin{aligned} ma_y &= m(\ddot{y} + ru_x) = F_{yf} + F_{yr}, \\ \mathbf{I}_z \dot{r} &= aF_{yf} - bF_{yr}, \end{aligned} \quad (5.7)$$

where

- i)  $F_{yf}, F_{yr}$ : the lateral tire forces of the front and rear wheels,
- ii)  $a_y$ : lateral acceleration;  $y$ : lateral translation;  $r$ : yaw rate;
- iii)  $u_x$ : longitudinal velocity;  $m$ : mass of vehicle;  $\mathbf{I}_z$ : inertia of the vehicle;
- iv)  $a, b$ : the distances of the front and rear tires from the c.g. of the vehicle.

Further, authors of [114] represented the lateral tire forces in the following form:

$$F_y = c_1\alpha - c_2\alpha^2 \operatorname{sgn}(\alpha) + c_3\alpha^3, \quad (5.8)$$

where  $\alpha$  denotes the tire slip angle, while  $c_1, c_2$  and  $c_3$  represent the coefficients of the tire form model. The slip angles of front tires and rear tires (i.e.,  $\alpha_f$  and  $\alpha_r$ ) are expressed as follows:

$$\begin{aligned} \alpha_f &= \delta - \left( \beta + \frac{ra}{u_x} \right), \\ \alpha_r &= - \left( \beta - \frac{rb}{u_x} \right), \end{aligned} \quad (5.9)$$

where  $\beta$  and  $\delta$  indicate the vehicle slip angle and the steering angle, respectively.

Therefore, through the utilisation of (5.7), (5.8) and (5.9), The nonlinear vehicle model is represented in the form of (3.68) with the subsequent parameters:

$$\begin{aligned} x &= \begin{bmatrix} \alpha_f \\ \alpha_r \end{bmatrix}, u = \begin{bmatrix} \delta \\ \dot{\delta} \\ \dot{a}_y \end{bmatrix}, A = \begin{bmatrix} -\left(\frac{u_x}{a+b} + \frac{a^2 c_{1f}}{I_z u_x}\right) & \left(\frac{u_x}{a+b} + \frac{abc_{1r}}{I_z u_x}\right) \\ -\left(\frac{u_x}{a+b} - \frac{abc_{1f}}{I_z u_x}\right) & \left(\frac{u_x}{a+b} - \frac{b^2 c_{1r}}{I_z u_x}\right) \end{bmatrix}, B = \begin{bmatrix} \frac{u_x}{a+b} & 1 & \frac{-1}{u_x} \\ \frac{u_x}{a+b} & 0 & \frac{-1}{u_x} \end{bmatrix}, G = \begin{bmatrix} \frac{a^2}{I_z u_x} & -\frac{ab}{I_z u_x} \\ -\frac{ab}{I_z u_x} & \frac{b^2}{I_z u_x} \end{bmatrix}, \\ C &= \begin{bmatrix} -\frac{u_x}{a+b} & \frac{u_x}{a+b} \\ \frac{c_{1f}}{m} & \frac{c_{1r}}{m} \end{bmatrix}, H = \begin{bmatrix} 0 & 0 \\ \frac{-1}{m} & \frac{-1}{m} \end{bmatrix}, E = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

$f(x) = \begin{bmatrix} -\eta(\alpha_f) \\ -\eta(\alpha_r) \end{bmatrix}$  and  $h(x) = \begin{bmatrix} -\eta(\alpha_f) \\ -\eta(\alpha_r) \end{bmatrix}$  are the nonlinearities present in the dynamics and outputs, respectively, along with

$$\eta(\zeta) = -c_2\zeta^2 \operatorname{sgn}(\zeta) + c_3\zeta^3.$$

The parameters inside the model are showcased in Table 5.4. Let us consider that both dynamics and measurements are affected by the Gaussian noise ( $\omega \rightsquigarrow (0, 1)$ ).

### 5.3.2 Employing ISS-based observer

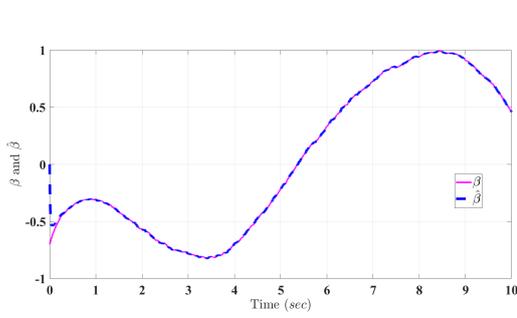
In this subsection, the proposed observer is utilised for the estimation of the slip angle of a nonlinear autonomous vehicle model.

The proposed LMI (3.122) is solved in the MATLAB toolbox for the above model, and we achieve:

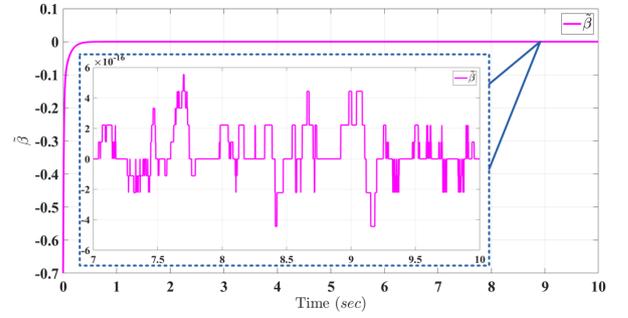
$$\delta = 1.969 \times 10^{-20}, \quad \gamma = 9.7954 \times 10^{-15} \text{ and } L = \begin{bmatrix} 0.0535 & 0.9465 \\ 0.3580 & 0.6420 \end{bmatrix}.$$

Table 5.4: Parameters of vehicle model

No.	Parameter	Description	Value	unit
1	$m$	mass of the vehicle	1573	kg
2	$\mathbf{I}_z$	Inertia of the vehicle	2873	$\text{kg} \cdot \text{m}^2$
3	$a$	distance of front tires from c.g. of the vehicle	1.1	m
4	$b$	distance of rear tires from c.g. of the vehicle	1.58	m
5	$u_x$	longitudinal velocity	10	$\text{m} \cdot \text{s}^{-1}$
6	$r$	Yaw rate	0.8	$\text{rad} \cdot \text{s}^{-1}$
7	$c_2, c_3$	coefficient of tire models	8000	-
8	$c_{1f}, c_{1r}$	coefficient of tire models	80000	-



(a) Plot of  $\beta$  and  $\hat{\beta}$



(b) Plot of  $\hat{\beta} - \beta$

Figure 5.4: Estimation of Slip angle

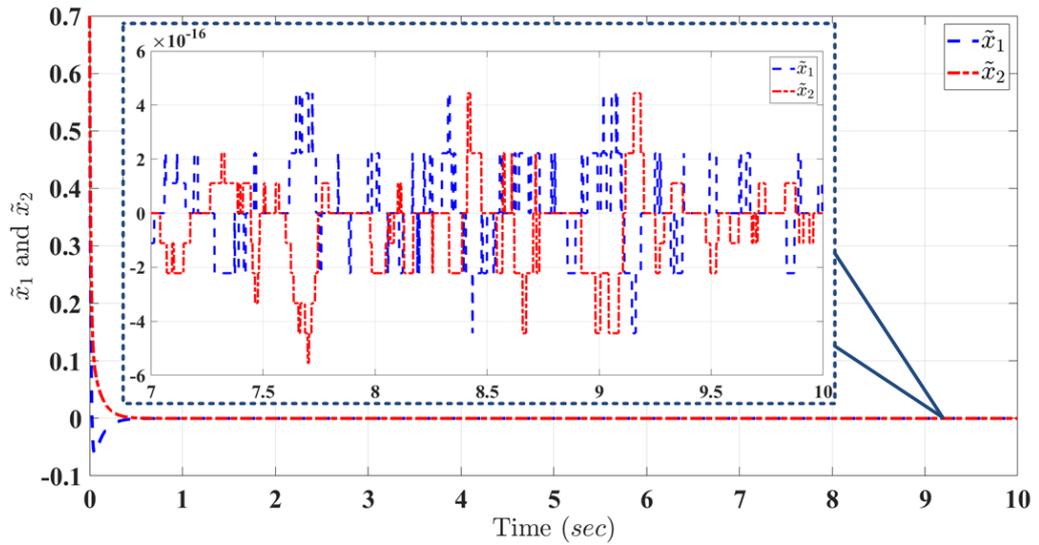


Figure 5.5: Graph of estimation error ( $\tilde{x}$ )

The observer (3.71) is deployed in the MATLAB environment for the estimation of the slip angle of a nonlinear autonomous vehicle. Figure 5.5 depicts the plot of the estimation error ( $\tilde{x}$ ) obtained from the observer. It highlights the efficiency of the proposed observer which provides accurate state estimation with the optimal noise attenuation. Further, the graphs of the slip angle estimation ( $\beta$ ) and its error are showcased in Figure 5.4. The estimation accuracy is described in Figure 5.4a and 5.4b. All these figures emphasise the precise estimation of the slip angle of the vehicle with efficient noise compensation. Therefore, the performance of the developed LMI-based observer is validated.

## 5.4 Transformation-based observer for vehicle tracking model

The lateral and longitudinal position of an autonomous vehicle plays a vital role in vehicle tracking problems. The model proposed by the authors of [114] is widely used in the estimation of these components. The several observer methodologies for vehicle tracking problems are developed in [123], [124], [125] and [15]. The objective of this section is to propose the transformation-based observer for the purpose of the state estimation of a nonlinear vehicle model.

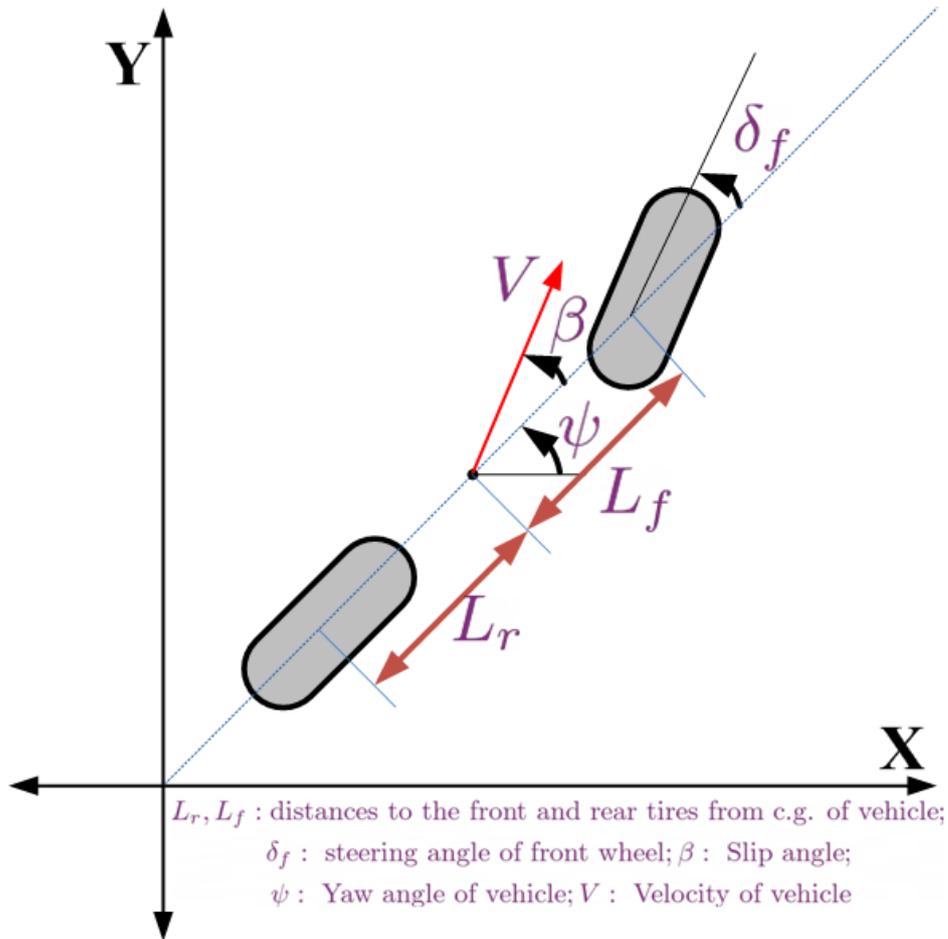


Figure 5.6: Vehicle motion model

### 5.4.1 Vehicle model

In this section, a bicycle model from [15] for each tracked vehicle is used for observer design, and it is showcased in Figure 5.6.

Let us define the state vector of the vehicle model as

$$\eta = [X \quad Y \quad \psi \quad \delta_f]^\top,$$

where  $X$  and  $Y$  denote a relative longitudinal position and relative lateral position of the vehicle, respectively.  $\psi$  and  $\delta_f$  represent the vehicle's yaw angle and steering angle of the front wheel, respectively. The derivative of the steering angle is presumed to be zero. Thus, the model dynamics are illustrated as follows:

$$\dot{\eta} = \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{\psi} \\ \dot{\delta}_f \end{bmatrix} = \begin{bmatrix} V \cos(\psi) \\ V \sin(\psi) \\ \frac{V}{L_f + L_r} \tan(\delta_f) \\ 0 \end{bmatrix}, \quad (5.10)$$

where  $V$  stands for the velocity of the vehicle. Additionally,  $L_f$ ,  $L_r$  are used to symbolise the distances to the front and rear tires from the vehicle's centre of gravity, respectively. Further, the output of the vehicle is expressed as:

$$\eta_y = \begin{bmatrix} X \\ Y \end{bmatrix}. \quad (5.11)$$

Here, the process dynamics are nonlinear while the output equations are linear.

The state-space model represented in (5.10) and (5.11) is analogous to the one illustrated in (2.33). Hence, one can easily deploy the observer based on the proposed LMIs (2.55) (2.99) and (2.100) for the purpose of vehicle tracking. However, these aforementioned LMIs provide an infeasible solution for this example.

#### Objective

The primary goal of this section is to develop a [transformation-based observer](#) for vehicle tracking problems.

### 5.4.2 Observer development

One of the possible solutions to achieve the earlier-stated objective is to utilise a transformation-based observer. With the help of a nonlinear transformation, the existing system is transformed into a new form of system which does not contain any nonmonotonous terms. After transformation, the proposed LMI-based observer can be implemented on the transformed model to estimate the state of systems. The generalised schematic representation of this approach is described in Figure 5.7. One can refer to Section 1.3.1 for more details about this methodology. The transformation  $\chi : \eta \rightarrow z$  of class  $C^2$  for the autonomous vehicle model (5.10) is defined as

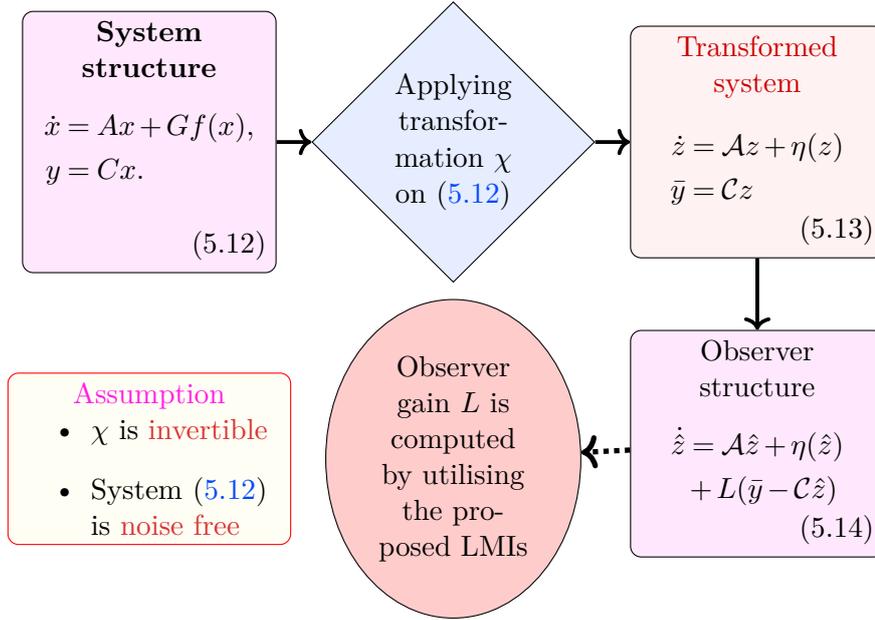


Figure 5.7: Graphical depiction of the proposed transformation-based observer

follows:

$$\begin{aligned}
 z_1 &= \eta_{y_1} = X \\
 z_2 &= \eta_{y_2} = Y \\
 z_3 &= \dot{z}_1 = \dot{X} = V \cos(\psi) \\
 z_4 &= \dot{z}_2 = \dot{Y} = V \sin(\psi) \\
 z_5 &= \dot{z}_3 = \ddot{X} = -V \sin(\psi) \dot{\psi} = -\alpha \times v \sin(\psi) \\
 z_6 &= \dot{z}_4 = \ddot{Y} = V \cos(\psi) \dot{\psi} = \alpha \times V \cos(\psi),
 \end{aligned} \tag{5.15}$$

where  $\alpha = \frac{V}{L_f + L_r} \tan(\delta_f)$ .

Further, through utilisation of (5.15),  $\alpha$  is expressed in the subsequent form:

$$\alpha = \frac{1}{V^2} (-z_4 z_5 + z_3 z_6) \tag{5.16}$$

Therefore, the transformed model for an autonomous vehicle system (5.10) is illustrated as follows:

$$\begin{aligned}
 \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} &= \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} \\
 \begin{pmatrix} \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} &= \begin{pmatrix} z_5 \\ z_6 \end{pmatrix} \\
 \begin{pmatrix} \dot{z}_5 \\ \dot{z}_6 \end{pmatrix} &= \left( \frac{-z_4 z_5 + z_3 z_6}{v^2} \right)^2 \begin{pmatrix} -z_3 \\ -z_4 \end{pmatrix} \\
 y &= \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.
 \end{aligned} \tag{5.17}$$

The state-space representation of (5.17) is given by

$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \dot{z}_5 \\ \dot{z}_6 \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A_z} \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix}}_z + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -\frac{1}{(V^2)^2} & 0 \\ 0 & -\frac{1}{(V^2)^2} \end{bmatrix}}_{G_z} \underbrace{\begin{bmatrix} (-z_4 z_5 + z_3 z_6)^2 z_3 \\ (-z_4 z_5 + z_3 z_6)^2 z_4 \end{bmatrix}}_{f(z)}, \\ y &= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}}_{C_z} z \end{aligned} \quad (5.18)$$

The system showcased in (5.18) is in the form of (2.33). The parameters corresponding to the nonlinearities of (5.18) are as follows:

$$H_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}; \quad H_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \quad \text{Hence, } m = 2 \text{ and } \bar{n} = 4.$$

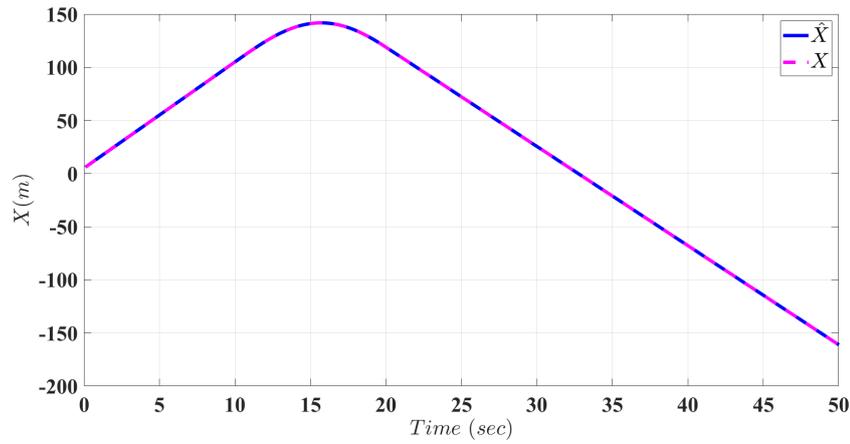
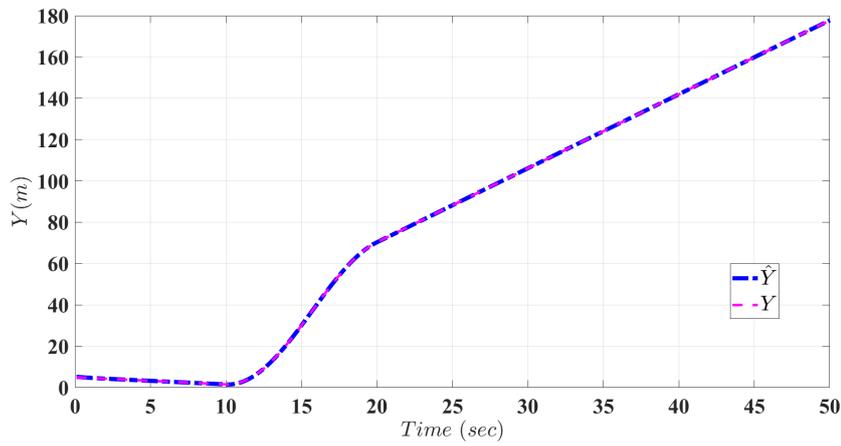
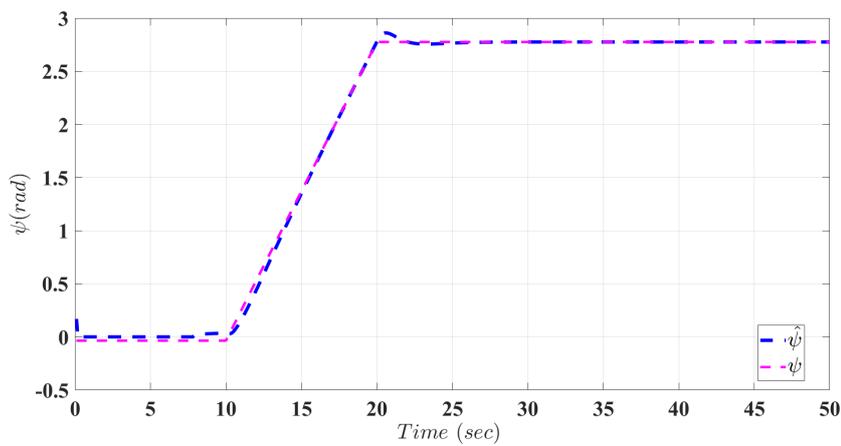
Since yaw angle  $\psi$  vary as  $0 \leq \psi \leq \pi$ ,  $z_3$  and  $z_4$  hold  $-V \leq z_3 \leq V$  and  $0 \leq z_4 \leq V$ , respectively. However, variation in  $z_5$  and  $z_6$  depends on the values of both  $\psi$  and  $\delta_f$ . It is assumed that the slip angle of the front wheel fulfils  $-10 \leq \delta_f \leq 10$ . It infers that the partial derivatives of nonlinearities are bounded. Further, the gain matrix  $L$  of the observer (2.35) is computed by solving LMI (2.99) in the YALMIP toolbox, which is as follows:

$$L = \begin{bmatrix} 36.9026 & -0.0575 \\ -0.0576 & 36.9140 \\ 68.0788 & -0.0338 \\ -0.0339 & 68.0848 \\ 34.9946 & -0.1109 \\ -0.1126 & 35.0030 \end{bmatrix}.$$

The proposed observer (2.35) with the aforementioned gain matrix is implemented in MATLAB, and the state variables of the transformed model are estimated, i.e.  $z$  vector. However, the main purpose of this estimation method is to compute the state of the vehicle model, that is,  $x$ ,  $y$ ,  $\psi$  and  $\delta$ . From the estimated vector  $\hat{z}$ , the vehicle trajectories are achieved through the utilisation of the ensuing inverse transformation:

$$\begin{aligned} \hat{x}_1 &= z_1, \\ \hat{x}_2 &= z_2, \\ \hat{x}_3 &= \tan^{-1}\left(\frac{z_4}{z_3}\right), \\ \hat{x}_4 &= \tan^{-1}\left(\frac{l_f + l_r}{V^3}\right)(z_3 z_6 - z_4 z_5). \end{aligned} \quad (5.19)$$

Figure 5.8 depicts the plot of estimated state  $\hat{x}_1$  obtained from the proposed observer and actual state  $x_1$  of the vehicle model. Whereas, the graph of  $\hat{x}_2$  and  $x_2$  is portrayed in Figure 5.9. The estimation of the yaw angle of the autonomous vehicle is shown in Figure 5.10. All these figures highlight the accuracy of the proposed transformation-based observer. Thus, the performance of the developed observer is validated.

Figure 5.8: Estimation of relative longitudinal position ( $X$ ) of a vehicleFigure 5.9: Estimation of relative lateral position ( $Y$ ) of a vehicleFigure 5.10: Estimation of yaw angle ( $\psi$ ) of a vehicle

## 5.5 Conclusion

In this chapter, several problems on the topic of autonomous vehicles are addressed. First, the proposed LMI-based observer of Chapter 3 is deployed for the longitudinal state estimation of a third-order nonlinear autonomous vehicle model. The problem of the slip angle estimation of autonomous vehicles is tackled by utilising the newly derived ISS-based nonlinear observer of Chapter 3. Later on, the vehicle tracking problem is solved by integrating transformation-based observer methodology and the proposed LMI of Chapter 2. Hence, with these applications, the performance of the proposed LMI-based observers is verified and analysed.





---

## Conclusion and perspectives

*“Out of every end, there is a new beginning waiting to be written”*

---

*Unknown*

This thesis is focused on the topic of nonlinear observer design using LMI relaxation techniques. In the first part, the authors have primarily provided a brief summary of the fundamental notions utilised in the development of nonlinear observers and several observer design methodologies. In particular, some important concepts of observability and stability are illustrated. In addition to this, some mathematical tools required for the observer design were recalled. All these retrieved fundamentals were used in the subsequent parts of this thesis for the formulation of the new LMI conditions.

Further, a few novel LMI-based observer approaches have been proposed in this thesis. Through judicious utilisation of the newly defined matrix multipliers, namely (2.82) and (2.83); applying Young inequalities (A.9) and (A.10); and incorporating the reformulated Lipschitz property, as outlined in Definition 3 of Appendix A.1, new LMI conditions are established. These obtained LMIs provide better feasibility for larger Lipschitz constant than those presented in the literature, which is demonstrated through a numerical example. The inclusion of a generalised matrix multiplier in the LMI formulation introduces extra decision variables within these conditions, enhancing degrees of freedom and consequently improving their feasibility. The performance of the developed LMI-based observers is validated by using numerical examples.

Later on, the established LMI conditions are deployed in the case of  $\mathcal{H}_\infty$  nonlinear observer design. The obtained LMIs are less conservative than those showcased in [12–14], and this is demonstrated through numerical examples. Furthermore, this technique is infused with the recently proposed Hilbert projection-based Luenberger observer of [23] for the state estimation of the disturbance-affected non-global Lipschitz system. Later in this thesis, we have designed the observer-based stabilization strategy for nonlinear systems using the aforementioned LMI approach.

In the last part of the thesis, the applicability of the established LMI-based observer in the domain of autonomous vehicles is demonstrated. We deployed the proposed LMI-based observer of Chapter 3 to estimate the longitudinal state of a third-order nonlinear autonomous vehicle model. The authors have employed the newly derived ISS-based nonlinear observer of Chapter 3 for the purpose of the slip angle estimation of autonomous vehicles. Later on, the vehicle tracking problem is solved by integrating transformation-based observer methodology and the proposed LMI of Chapter 2. Hence, with these applications, the performance of the proposed LMI-based observers is verified and analysed.

## Research prospects

One of the major contributions of this thesis is the development of the matrix-multiplier-based LMI methods for nonlinear observer design. Considering the importance of nonlinear systems and LMIs in control systems, the proposed approaches open up new research possibilities in various aspects of control engineering. The potential future tasks related to the new LMI techniques from the author's point of view are listed as follows:

1. **Nonlinear observer design for discrete-time Systems:** The utilisation of reformulated Lipschitz properties of nonlinearities, Young inequalities, and the proposed matrix multipliers in establishing nonlinear observers for continuous-time systems has yielded enhanced LMI conditions. We intend to apply a similar approach to develop observers for nonlinear discrete-time Lipschitz systems, aiming to derive less conservative LMI conditions compared to existing ones. Subsequently, we will employ the proposed observer to accomplish tasks such as output tracking or stabilization in these systems.
2. **Exploiting the proposed observers for uncertain systems:** In complex nonlinear systems, uncertainty present in dynamics or outputs is one of the critical obstacles during the state estimation process. As a result, control systems researchers have shown a notable level of interest in this area. The authors intend to use the established LMI approach in this area, which may result in efficient state estimation tools for these types of systems.
3. **Estimation of exogenous signals, such as faults, disturbances, and cyber-attacks, present in the nonlinear systems:** The continuous growth of modern-day technologies has resulted in nonlinear complex systems becoming more vulnerable to external influences, such as noise, faults, and cyber-attacks. In addition to monitoring the states of these systems, it is crucial to gather data on these exogenous signals to mitigate the risk of system failures. The authors had planned to solve this problem with the incorporation of the existing unknown input observer methods with the proposed LMI techniques.

In addition to this, as the authors mentioned earlier, the developed matrix-multipliers-based LMI approach can be deployed in other problems of control domains, such as observer design for delayed systems, observer design of LPV systems, adaptive observer design for parameter estimations, and so on.

# Tools used in nonlinear observer design

In observer design, the researchers used many mathematical tools to develop significant observers. Some of these tools are illustrated in this section.

## A.1 Lipschitz property

One of the common tools used to deal with nonlinear functions in observer design problems is **Lipschitz property of nonlinear functions**. It was initially proposed by the German mathematician Rudolf Lipschitz. Later on, authors of [33] demonstrated that it is very essential to show the existence and uniqueness of the solution of (1.1) or (1.31).

### Definition A.1: Lipschitz condition [33]

Let  $f(x, u)$  be piece-wise continuous function and holds:

$$\|f(x, u) - f(x_1, u)\| \leq \gamma_f(x - x_1), \quad \forall x, x_1 \in \mathbb{R}^n, x \neq x_1, \&\forall u \in \mathbb{R}^s. \quad (\text{A.1})$$

where  $\gamma_f$  is a constant. Inequality (A.1) is known as **Lipschitz condition** and constant  $\gamma_f$  is called as a **Lipschitz constant** of function  $f$ .

The Lipschitz property plays a vital role in nonlinear observer design. Various approaches for the development of observers has been presented in [66], [12], [32], [33] and so on. Further, authors of [11] proposed a less conservative Lipschitz condition which is described as:

$$\|f(x, u) - f(x_1, u)\| \leq \mathcal{T}(x - x_1), \quad \forall x, x_1 \in \mathbb{R}^n, x \neq x_1, \&\forall u \in \mathbb{R}^s, \quad (\text{A.2})$$

where  $\mathcal{T}$  is a sparsely populated matrix, and  $\mathcal{T}(x - x_1) \leq \gamma_f(x - x_1)$ . The use of such a less conservative Lipschitz condition enhances the existing observer conditions [11].

Furthermore, the existing Lipschitz condition is reformulated in [21] to derive a more efficient observer condition than the one that exists in the literature. Before introducing the reformulated Lipschitz property, let us consider the following definition:

**Definition A.2: Auxiliary vector notation ( $X^{Y_i}$ ) [21]**

Consider two vectors:

$$X = (x_1 \ \cdots \ x_n)^\top \quad (\text{A.3})$$

$$Y = (y_1 \ \cdots \ y_n)^\top. \quad (\text{A.4})$$

Then, for all  $i = 0, \dots, n$ , an auxiliary vector  $X^{Y_i} \in \mathbb{R}^n$  corresponding to  $X$  and  $Y$  can be defined as,

$$X^{Y_i} = \begin{cases} (y_1 \ \cdots \ y_i \ x_{i+1} \ \cdots \ x_n)^\top, & \text{for } i = 1, \dots, n \\ X, & \text{for } i = 0. \end{cases} \quad (\text{A.5})$$

**Definition A.3: Reformulated Lipschitz property [21]**

For the given function  $f$  as defined in the Definition 1, the following statement is true:

For all,  $i, j = 1, \dots, n$ , there exists functions  $f_{ij} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  and constants  $f_{ij_{\min}}$  and  $f_{ij_{\max}}$  such that  $\forall X, Y \in \mathbb{R}^n$ ,

$$f(X) - f(Y) = \sum_{i=1}^n \sum_{j=1}^n f_{ij} \mathcal{H}_{ij}(X - Y), \quad (\text{A.6})$$

where  $\mathcal{H}_{ij} = e_n(i)e_n^\top(j)$ , and  $f_{ij} \triangleq f_{ij}(X_i^{Y_{i,j-1}}, X_i^{Y_{i,j}})$ . The function  $f_{ij}(X^{Y_{j-1}}, X^{Y_j})$  is defined as follows:

$$f_{ij}(X^{Y_{j-1}}, X^{Y_j}) = \begin{cases} 0, & \text{for } x_j = y_j \\ \frac{f(X^{Y_{j-1}}) - f(X^{Y_j})}{X^{Y_{j-1}} - X^{Y_j}}, & \text{for } x_j \neq y_j. \end{cases} \quad (\text{A.7})$$

The functions  $f_{ij}(\cdot)$  are globally bounded as follows:

$$f_{ij_{\min}} \leq f_{ij} \leq f_{ij_{\max}} \quad (\text{A.8})$$

## A.2 Young Relation (Completion of the Squares)

For any two vectors  $X, Y \in \mathbb{R}^n$  and a matrix  $Z = Z^\top > 0 \in \mathbb{R}^{n \times n}$ , the following matrix inequality holds:

$$X^\top Y + Y^\top X \leq X^\top Z^{-1} X + Y^\top Z Y. \quad (\text{A.9})$$

The inequality (A.9) is known as Young inequality. The authors of [14] proposed a variant of Young inequality which is given by

$$X^\top Y + Y^\top X \leq \frac{1}{2}(X + ZY)^\top Z^{-1}(X + ZY). \quad (\text{A.10})$$

Let us expand  $\frac{1}{2}(X + ZY)^\top Z^{-1}(X + ZY)$  as follows:

$$\frac{1}{2}(X + ZY)^\top Z^{-1}(X + ZY) = \frac{1}{2}(X^\top Z^{-1}X + Y^\top ZY) + \frac{1}{2}(X^\top Y + Y^\top X)$$

By implementing the inequality (A.9) on  $\frac{1}{2}(X^\top Y + Y^\top X)$ , the following inequality is obtained:

$$\frac{1}{2}(X^\top Y + Y^\top X) \leq \frac{1}{2}(X^\top Z^{-1}X + Y^\top ZY) + \frac{1}{2}(X^\top Z^{-1}X + Y^\top ZY).$$

Further, it leads to

$$\frac{1}{2}(X + ZY)^\top Z^{-1}(X + ZY) \leq \frac{1}{2}(X^\top Z^{-1}X + Y^\top ZY) + \frac{1}{2}(X^\top Z^{-1}X + Y^\top ZY).$$

Hence,

$$X^\top Y + Y^\top X \leq \frac{1}{2}(X + ZY)^\top Z^{-1}(X + ZY) \leq (X^\top Z^{-1}X + Y^\top ZY). \quad (\text{A.11})$$

Both Inequalities (A.9) and (A.9) are used in various observer design approaches, especially, in LMI-based methods, for example, [71], [14] and [22].

### A.3 Schur lemma

#### Definition A.4: Schur's compliment [126]

If there exist  $A = A^\top \in \mathbb{R}^{n \times n}$ ,  $B = B^\top \in \mathbb{R}^{m \times m}$  and  $C \in \mathbb{R}^{n \times m}$  then the following statements are equivalent.

1.  $M = \begin{bmatrix} A & C \\ C^\top & B \end{bmatrix} < 0$
2.  $A - C^\top B^{-1}C < 0$  and  $B < 0$
3.  $B - C^\top A^{-1}C < 0$  and  $A < 0$

The statements (2) and (3) are known as **Schur's compliment** of a block matrix  $M$ .

Issai Schur, who utilized the Schur complement to prove Schur's lemma, was named after it. The Schur complement is a key element in the design of LMI-based observers.

### A.4 Linear Matrix Inequalities

This section aims to provide an overview of a fundamental mathematical tool used in control system engineering. This tool is known as **linear matrix inequality (LMI)**. It is introduced by the authors of [127] and [128]. Later on, some essential features of LMIs and their significance in the optimisation problem are illustrated in [18]. Let us define LMI as follows:

#### Definition A.5: LMI [18]

If there exist a matrix inequality  $\mathcal{F} : \mathbb{R}^m \rightarrow \mathbf{S}^n$ , in the variable  $x \in \mathbb{R}^m$  under the ensuing

form:

$$\mathcal{F}(x) = \mathcal{F}_0 + \sum_{i=1}^m x_i \mathcal{F}_i \leq 0, \quad (\text{A.12})$$

where  $x = [x_1 \ x_2 \ \dots \ x_m]^\top$  and  $\mathcal{F}_i \in \mathbf{S}^n \forall i \in \{1, \dots, m\}$ . Then, the matrix inequality  $\mathcal{F}$  is defined as a **linear matrix inequality (LMI)**.

From Definition 5, one can easily infer that  $\mathcal{F}$  is a special semi-definite constraint. In addition to this, it also implies that each element inside matrix  $\mathcal{F}$  is an affine function of  $x$ . The subsequent lemma establishes one of the vital characteristics of LMIs:

**Lemma A.1: Convexity of LMIs [129]**

The set of solutions obtained by solving an LMI (A.12) holds the property of convexity. In other words, the set of solutions to (A.12) is a convex set.

**Proof:** One can refer to [129] for the proof of the aforementioned lemma. □

Lemma 5.1 implies that an optimisation problem, which comprises an LMI and a convex objective function, is convex. It is one of the advantageous properties of LMI and is widely used in the control domain.



---

# Bibliography

## Bibliography

- [1] Zhong-Hua Pang, Wen-Cheng Luo, Guo-Ping Liu, and Qing-Long Han. Observer-based incremental predictive control of networked multi-agent systems with random delays and packet dropouts. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 68(1):426–430, 2021.
- [2] Woongsun Jeon, Zhenming Xie, Ali Zemouche, and Rajesh Rajamani. Simultaneous cyber-attack detection and radar sensor health monitoring in connected ACC vehicles. *IEEE Sensors Journal*, 21(14):15741–15752, 2020.
- [3] Brian DO Anderson and John B Moore. *Optimal filtering*. Courier Corporation, 2012.
- [4] D. Luenberger. An introduction to observers. *IEEE Transactions on Automatic Control*, 16(6):596–602, 1971.
- [5] R. Diversi, R. Guidorzi, and U. Soverini. Kalman filtering in extended noise environments. *IEEE Transactions on Automatic Control*, 50(9):1396–1402, 2005.
- [6] Henrique M. T. Menegaz, João Y. Ishihara, Geovany A. Borges, and Alessandro N. Vargas. A systematization of the unscented kalman filter theory. *IEEE Transactions on Automatic Control*, 60(10):2583–2598, 2015.
- [7] Ali Zemouche, Fan Zhang, Frédéric Mazenc, and Rajesh Rajamani. High-gain nonlinear observer with lower tuning parameter. *IEEE Transactions on Automatic Control*, 64(8):3194–3209, 2018.
- [8] T. Ahmed-Ali and F. Lamnabhi-Lagarrigue. Sliding observer-controller design for uncertain triangular nonlinear systems. *IEEE Transactions on Automatic Control*, 44(6):1244–1249, 1999.

- 
- [9] Salim Ibrir, Wen Fang Xie\*, and Chun-Yi Su. Observer-based control of discrete-time lipschitzian non-linear systems: application to one-link flexible joint robot. *International Journal of Control*, 78(6):385–395, 2005.
- [10] Salim Ibrir. Circle-criterion approach to discrete-time nonlinear observer design. *Automatica*, 43(8):1432–1441, 2007.
- [11] G. Phanomchoeng and R. Rajamani. Observer design for lipschitz nonlinear systems using riccati equations. In *Proceedings of the 2010 American Control Conference*, pages 6060–6065, 2010.
- [12] Ali Zemouche and Mohamed Boutayeb. A unified  $\mathcal{H}_\infty$  adaptive observer synthesis method for a class of systems with both Lipschitz and monotone nonlinearities. *Systems & Control Letters*, 58(4):282–288, 2009.
- [13] Michelle Chong, Romain Postoyan, Dragan Nešić, Levin Kuhlmann, and Andrea Varsavsky. A robust circle criterion observer with application to neural mass models. *Automatica*, 48(11):2986–2989, 2012.
- [14] Ali Zemouche, Rajesh Rajamani, Gridsada Phanomchoeng, Boulaïd Boukroune, Hugues Rafaralahy, and Michel Zasadzinski. Circle criterion-based  $\mathcal{H}_\infty$  observer design for lipschitz and monotonic nonlinear systems—enhanced LMI conditions and constructive discussions. *Automatica*, 85:412–425, 2017.
- [15] Rajesh Rajamani, Woongsun Jeon, Hamidreza Movahedi, and Ali Zemouche. On the need for switched-gain observers for non-monotonic nonlinear systems. *Automatica*, 114:108814, 2020.
- [16] Xiao-Heng Chang and Guang-Hong Yang. New results on output feedback  $\mathcal{H}_\infty$  control for linear discrete-time systems. *IEEE Transactions on Automatic Control*, 59(5):1355–1359, 2013.
- [17] Ali Zemouche, Rajesh Rajamani, Hieu Trinh, and Michel Zasadzinski. A new LMI based  $\mathcal{H}_\infty$  observer design method for Lipschitz nonlinear systems. In *2016 European Control Conference (ECC)*, pages 2011–2016. IEEE, 2016.
- [18] Stephen Boyd, Laurent El Ghaoui, Eric Feron, and Venkataramanan Balakrishnan. *Linear matrix inequalities in system and control theory*. SIAM, 1994.
- [19] Sekhar Raghavan and J Karl Hedrick. Observer design for a class of nonlinear systems. *International Journal of Control*, 59(2):515–528, 1994.
- [20] A Alessandri. Design of observers for Lipschitz nonlinear systems using LMI. *IFAC Proceedings Volumes*, 37(13):459–464, 2004.
- [21] Ali Zemouche and Mohamed Boutayeb. On LMI conditions to design observers for Lipschitz nonlinear systems. *Automatica*, 49(2):585–591, 2013.
- [22] K Chaib, Draa, A Zemouche, M Alma, H Voos, and M Darouach. A discrete-time nonlinear state observer for the anaerobic digestion process. *International Journal of Robust and Nonlinear Control*, 29(5):1279–1301, 2019.

- 
- [23] Ali Zemouche and Rajesh Rajamani. Observer design for non-globally lipschitz nonlinear systems using hilbert projection theorem. *IEEE Control Systems Letters*, 6:2581–2586, 2022.
- [24] A Zemouche, Mohamed Zerrougui, Boulaïd Boukroune, Fazia Bedouhene, H Souley-Ali, and Michel Zasadzinski.  $\mathcal{H}_\infty$  observer-based stabilization for Lipschitz nonlinear systems. In *2016 European Control Conference (ECC)*, pages 2017–2022. IEEE, 2016.
- [25] Wen-Hua Chen. Disturbance observer based control for nonlinear systems. *IEEE/ASME transactions on mechatronics*, 9(4):706–710, 2004.
- [26] K. Chaib-Draa, A. Zemouche, M. Alma, H. Voos, and M. Darouach. A nonlinear observer-based trajectory tracking method applied to an anaerobic digestion process. *Journal of Process Control*, 75:120–135, 2019.
- [27] Mazen Alamir. Optimization based non-linear observers revisited. *International Journal of Control*, 72(13):1204–1217, 1999.
- [28] Gerta Zimmer. State observation by on-line minimization. *International Journal of Control*, 60(4):595–606, 1994.
- [29] Gildas Besançon. *Nonlinear observers and applications*, volume 363. Springer, 2007.
- [30] R. Hermann and A. Krener. Nonlinear controllability and observability. *IEEE Transactions on Automatic Control*, 22(5):728–740, 1977.
- [31] J Levine and R Marino. Nonlinear system immersion, observers, finite dimensional filters and applications. In *Analysis and Optimization of Systems: Proceedings of the Seventh International Conference on Analysis and Optimization of Systems, Antibes, June 25-27, 1986*, pages 869–880. Springer, 1986.
- [32] Angelo Alessandri. Observer design for nonlinear systems by using input-to-state stability. In *2004 43rd IEEE Conference on Decision and Control (CDC)(IEEE Cat. No. 04CH37601)*, volume 4, pages 3892–3897. IEEE, 2004.
- [33] Hassan K Khalil and Jessy W Grizzle. *Nonlinear systems*, volume 3. Prentice hall Upper Saddle River, NJ, 2002.
- [34] L. Grune, E.D. Sontag, and F. Wirth. On the equivalence between asymptotic and exponential stability, and between ISS and finite  $\mathcal{H}_\infty$  gain. In *Proceedings of the 38th IEEE Conference on Decision and Control (Cat. No.99CH36304)*, volume 2, pages 1220–1225 vol.2, 1999.
- [35] Eduardo D Sontag and Yuan Wang. On characterizations of the input-to-state stability property. *Systems & Control Letters*, 24(5):351–359, 1995.
- [36] Hyungbo Shim, Jin Heon Seo, and Andrew R Teel. Nonlinear observer design via passivation of error dynamics. *Automatica*, 39(5):885–892, 2003.
- [37] Bing Li, Zidong Wang, Lifeng Ma, and Hongjian Liu. Observer-based event-triggered control for nonlinear systems with mixed delays and disturbances: The input-to-state stability. *IEEE Transactions on Cybernetics*, 49(7):2806–2819, 2019.

- 
- [38] Hugh Durrant-Whyte et al. Introduction to estimation and the Kalman filter. *Australian Centre for Field Robotics*, 28(3):65–94, 2001.
- [39] David Luenberger. Observers for multivariable systems. *IEEE Transactions on Automatic Control*, 11(2):190–197, 1966.
- [40] Arthur J Krener and Alberto Isidori. Linearization by output injection and nonlinear observers. *Systems & Control Letters*, 3(1):47–52, 1983.
- [41] Driss Boutat, Frederic Kratz, and Jean-Pierre Barbot. Observability brunovsky normal form: Multi-output linear dynamical systems. In *2009 American Control Conference*, pages 1166–1170, 2009.
- [42] A Krener and W Respondek and AJ KRENER. Nonlinear observer with linearizable error dynamics. *SIAM Journal on Control and Optimization*, 23:197–216, 1985.
- [43] H Keller. Non-linear observer design by transformation into a generalized observer canonical form. *International Journal of Control*, 46(6):1915–1930, 1987.
- [44] Xiao-Hua Xia and Wei-Bin Gao. Nonlinear observer design by observer error linearization. *SIAM Journal on Control and Optimization*, 27(1):199–216, 1989.
- [45] J Rudolph and M Zeitz. A block triangular nonlinear observer normal form. *Systems & Control Letters*, 23(1):1–8, 1994.
- [46] M Hou and AC Pugh. Observer with linear error dynamics for nonlinear multi-output systems. *Systems & Control Letters*, 37(1):1–9, 1999.
- [47] Driss Boutat, G Zheng, Jean-Pierre Barbot, and Hassan Hammouri. Observer error linearization multi-output depending. In *Proceedings of the 45th IEEE Conference on Decision and Control*, pages 5394–5399. IEEE, 2006.
- [48] J Gauthier and G Bornard. Observability for any  $u(t)$  of a class of nonlinear systems. *IEEE Transactions on Automatic Control*, 26(4):922–926, 1981.
- [49] M Zeitz. Canonical forms for nonlinear systems. In *Nonlinear Control Systems Design 1989*, pages 33–38. Elsevier, 1990.
- [50] Antonio Tornambè. Use of asymptotic observers having-high-gains in the state and parameter estimation. In *Proceedings of the 28th IEEE Conference on Decision and Control*, pages 1791–1794. IEEE, 1989.
- [51] Guy Bornard and H Hammouri. A high gain observer for a class of uniformly observable systems. In *[1991] Proceedings of the 30th IEEE Conference on Decision and Control*, pages 1494–1496. IEEE, 1991.
- [52] Daniele Astolfi, Lorenzo Marconi, and Andrew Teel. Low-power peaking-free high-gain observers for nonlinear systems. In *2016 European Control Conference (ECC)*, pages 1424–1429. IEEE, 2016.
- [53] Shauying R Kou, David L Elliott, and Tzyh Jong Tarn. Exponential observers for nonlinear dynamic systems. *Information and control*, 29(3):204–216, 1975.

- 
- [54] Hassan Hammouri, Guy Bornard, and Krishna Busawon. High gain observer for structured multi-output nonlinear systems. *IEEE Transactions on automatic control*, 55(4):987–992, 2010.
- [55] Shauying R Kou, David L Elliott, and Tzyh Jong Tarn. Exponential observers for nonlinear dynamic systems. *Information and control*, 29(3):204–216, 1975.
- [56] Jean-Paul Gauthier, Hassan Hammouri, and Sami Othman. A simple observer for nonlinear systems applications to bioreactors. *IEEE Transactions on automatic control*, 37(6):875–880, 1992.
- [57] Angelo Alessandri and Anna Rossi. Time-varying increasing-gain observers for nonlinear systems. *Automatica*, 49(9):2845–2852, 2013.
- [58] Angelo Alessandri and Anna Rossi. Increasing-gain observers for nonlinear systems: Stability and design. *Automatica*, 57:180–188, 2015.
- [59] Daniele Astolfi and Lorenzo Marconi. A high-gain nonlinear observer with limited gain power. *IEEE Transactions on Automatic Control*, 60(11):3059–3064, 2015.
- [60] Daniele Astolfi, Lorenzo Marconi, Laurent Praly, and Andrew R Teel. Low-power peaking-free high-gain observers. *Automatica*, 98:169–179, 2018.
- [61] Ruipeng Xi, Huaguang Zhang, Shaoxin Sun, and Yingchun Wang. Reduced-order high-gain observer (rohgo)-based neural tracking control for random nonlinear systems with output delay. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 52(12):7507–7515, 2022.
- [62] Nicolas Boizot, Eric Busvelle, and Jean-Paul Gauthier. An adaptive high-gain observer for nonlinear systems. *Automatica*, 46(9):1483–1488, 2010.
- [63] Eric Bullinger and Frank Allgower. An adaptive high-gain observer for nonlinear systems. In *Proceedings of the 36th IEEE Conference on Decision and Control*, volume 5, pages 4348–4353. IEEE, 1997.
- [64] Jean-Paul Gauthier and Ivan AK Kupka. Observability and observers for nonlinear systems. *SIAM journal on control and optimization*, 32(4):975–994, 1994.
- [65] Hassan K Khalil and Laurent Praly. High-gain observers in nonlinear feedback control. *International Journal of Robust and Nonlinear Control*, 24(6):993–1015, 2014.
- [66] Rajesh Rajamani. Observers for lipschitz nonlinear systems. *IEEE transactions on Automatic Control*, 43(3):397–401, 1998.
- [67] Ian R Petersen and Christopher V Hollot. A riccati equation approach to the stabilization of uncertain linear systems. *Automatica*, 22(4):397–411, 1986.
- [68] Hamidreza Movahedi, Ning Tian, Huazhen Fang, and Rajesh Rajamani. Hysteresis compensation and nonlinear observer design for state-of-charge estimation using a nonlinear double-capacitor li-ion battery model. *IEEE/ASME Transactions on Mechatronics*, 2021.
- [69] Anirudh Nath, Dipankar Deb, and Rajeeb Dey. Robust observer-based adaptive control of blood glucose in diabetic patients. *International Journal of Control*, 94(11):3054–3067, 2021.

- 
- [70] Pasquale Palumbo, Pierdomenico Pepe, Simona Panunzi, and Andrea De Gaetano. Observer-based glucose control via subcutaneous insulin administration. *IFAC Proceedings Volumes*, 45(18):107–112, 2012.
- [71] Angelo Alessandri. Design of time-varying state observers for nonlinear systems by using input-to-state stability. In *2013 American Control Conference*, pages 280–285, 2013.
- [72] Hyungbo Shim and Daniel Liberzon. Nonlinear observers robust to measurement disturbances in an iss sense. *IEEE Transactions on Automatic Control*, 61(1):48–61, 2016.
- [73] A. Howell and J.K. Hedrick. Nonlinear observer design via convex optimization. In *Proceedings of the 2002 American Control Conference (IEEE Cat. No.CH37301)*, volume 3, pages 2088–2093 vol.3, 2002.
- [74] Murat Arcaç and Petar Kokotović. Nonlinear observers: a circle criterion design and robustness analysis. *Automatica*, 37(12):1923–1930, 2001.
- [75] M. Arcaç and P. Kokotovic. Observer-based control of systems with slope-restricted nonlinearities. *IEEE Transactions on Automatic Control*, 46(7):1146–1150, 2001.
- [76] Xingzhe Fan and Murat Arcaç. Observer design for systems with multivariable monotone nonlinearities. *Systems & Control Letters*, 50(4):319–330, 2003.
- [77] Behçet Açıkmeşe and Martin Corless. Observers for systems with nonlinearities satisfying incremental quadratic constraints. *Automatica*, 47(7):1339–1348, 2011.
- [78] A Farhangfar and RJ Shor. State observer design for a class of lipschitz nonlinear system with uncertainties. *IFAC-PapersOnLine*, 53(1):283–288, 2020.
- [79] Claudia Califano, Salvatore Monaco, and Dorothee Normand-Cyrot. On the observer design in discrete-time. *Systems & Control Letters*, 49(4):255–265, 2003.
- [80] FE Thau. Observing the state of non-linear dynamic systems. *International journal of control*, 17(3):471–479, 1973.
- [81] Lei Wang, Daniele Astolfi, Lorenzo Marconi, and Hongye Su. High-gain observers with limited gain power for systems with observability canonical form. *Automatica*, 75:16–23, 2017.
- [82] Pauline Bernard. *Observer design for nonlinear systems*, volume 479. Springer, 2019.
- [83] Woongsun Jeon, Ali Zemouche, and Rajesh Rajamani. Tracking of vehicle motion on highways and urban roads using a nonlinear observer. *IEEE/ASME transactions on mechatronics*, 24(2):644–655, 2019.
- [84] Francisco González de Cossío, Madiha Nadri, and Pascal Dufour. Observer design for nonlinear systems with output transformation. *IEEE Transactions on Automatic Control*, 65(12):5205–5219, 2020.
- [85] Gridsada Phanomchoeng, Rajesh Rajamani, and Damrongrit Piyabongkarn. Nonlinear observer for bounded jacobian systems, with applications to automotive slip angle estimation. *IEEE Transactions on Automatic Control*, 56(5):1163–1170, 2011.

- 
- [86] Jun Huang, Lin Yang, and Hieu Trinh. Robust control for incremental quadratic constrained nonlinear time-delay systems subject to actuator saturation. *Applied Mathematics and Computation*, 405:126271, 2021.
- [87] Heng Wang, Ali Zemouche, and Rajesh Rajamani. Nonlinear observer for electromagnetic position estimation using active current control. *Mechanical Systems and Signal Processing*, 167:108449, 2022.
- [88] Tapas Mazumdar. Generalized projection theorem and weak noncoercive evolution problems in hilbert space. *Journal of Mathematical Analysis and Applications*, 46(1):143–168, 1974.
- [89] Joachim Deutscher. Robust output regulation by observer-based feedforward control. *International Journal of Systems Science*, 48(4):795–804, 2017.
- [90] Mohsen Ekramian. Observer-based controller for lipschitz nonlinear systems. *International Journal of Systems Science*, 48(16):3411–3418, 2017.
- [91] Cuauhtemoc Acosta Lua, Bernardino Castillo-Toledo, Riccardo Cespi, and Stefano Di Genaro. Nonlinear observer-based active control of ground vehicles with non negligible roll dynamics. *International Journal of Control, Automation and Systems*, 14:743–752, 2016.
- [92] Chang-Hua Lien. Robust observer-based control of systems with state perturbations via lmi approach. *IEEE Transactions on Automatic Control*, 49(8):1365–1370, 2004.
- [93] Houria Kheloufi, Ali Zemouche, Fazia Bedouhene, and Mohamed Boutayeb. On lmi conditions to design observer-based controllers for linear systems with parameter uncertainties. *Automatica*, 49(12):3700–3704, 2013.
- [94] Hassan K Khalil. A note on the robustness of high-gain-observer-based controllers to unmodeled actuator and sensor dynamics. *Automatica*, 41(10):1821–1824, 2005.
- [95] Mario Di Ferdinando and Pierdomenico Pepe. On emulation of observer-based stabilizers for nonlinear systems. In *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, pages 6738–6743. IEEE, 2017.
- [96] Houria Kheloufi, Ali Zemouche, Fazia Bedouhene, and H Souley-Ali. A robust observer-based stabilization method for systems with uncertain parameters and Lipschitz nonlinearities. *International Journal of Robust and Nonlinear Control*, 26(9):1962–1979, 2016.
- [97] Noussaiba Gasmi, Mohamed Boutayeb, Assem Thabet, and Mohamed Aoun. Enhanced LMI conditions for observer-based  $\mathcal{H}_\infty$  stabilization of lipschitz discrete-time systems. *European Journal of Control*, 44:80–89, 2018.
- [98] Mohsen Ekramian, Saeed Hosseinnia, and Farid Sheikholeslam. General framework in designing luenberger-like non-linear observer. *IET Control Theory & Applications*, 7(2):253–260, 2013.
- [99] J. Lofberg. Yalmip : a toolbox for modeling and optimization in matlab. In *2004 IEEE International Conference on Robotics and Automation (IEEE Cat. No.04CH37508)*, pages 284–289, 2004.

- 
- [100] M Nadeem Ahangar, Qasim Z Ahmed, Fahd A Khan, and Maryam Hafeez. A survey of autonomous vehicles: Enabling communication technologies and challenges. *Sensors*, 21(3):706, 2021.
- [101] Johannes Betz, Hongrui Zheng, Alexander Liniger, Ugo Rosolia, Phillip Karle, Madhur Behl, Venkat Krovi, and Rahul Mangharam. Autonomous vehicles on the edge: A survey on autonomous vehicle racing. *IEEE Open Journal of Intelligent Transportation Systems*, 3:458–488, 2022.
- [102] Chris Urmson, Joshua Anhalt, Drew Bagnell, Christopher Baker, Robert Bittner, MN Clark, John Dolan, Dave Duggins, Tugrul Galatali, Chris Geyer, et al. Autonomous driving in urban environments: Boss and the urban challenge. *Journal of field Robotics*, 25(8):425–466, 2008.
- [103] Avery Hartmans. How google’s self-driving car project rose from a crazy idea to a top contender in the race toward a driverless future. *Business Insider*. Available online: <https://www.businessinsider.com/google-driverless-car-history-photos-2016-10> (accessed on December 2021), 2016.
- [104] S Oâ€™Kane. Uber debuts a new self-driving car with more fail-safes. *The Verge*, 12, 2019.
- [105] Vicente Milanés, Steven E Shladover, John Spring, Christopher Nowakowski, Hiroshi Kawazoe, and Masahide Nakamura. Cooperative adaptive cruise control in real traffic situations. *IEEE Transactions on intelligent transportation systems*, 15(1):296–305, 2013.
- [106] Bart Van Arem, Cornelie JG Van Driel, and Ruben Visser. The impact of cooperative adaptive cruise control on traffic-flow characteristics. *IEEE Transactions on intelligent transportation systems*, 7(4):429–436, 2006.
- [107] S.A. Nobe and Fei-Yue Wang. An overview of recent developments in automated lateral and longitudinal vehicle controls. In *2001 IEEE International Conference on Systems, Man and Cybernetics. e-Systems and e-Man for Cybernetics in Cyberspace (Cat.No.01CH37236)*, volume 5, pages 3447–3452 vol.5, 2001.
- [108] Shengbo Eben Li, Yang Zheng, Keqiang Li, Le-Yi Wang, and Hongwei Zhang. Platoon control of connected vehicles from a networked control perspective: Literature review, component modeling, and controller synthesis. *IEEE Transactions on Vehicular Technology*, 2017.
- [109] Shengbo Eben Li, Xiaohui Qin, Keqiang Li, Jianqiang Wang, and Boyuan Xie. Robustness analysis and controller synthesis of homogeneous vehicular platoons with bounded parameter uncertainty. *IEEE/ASME Transactions on Mechatronics*, 22(2):1014–1025, 2017.
- [110] Shengbo Eben Li, Yang Zheng, Keqiang Li, Le-Yi Wang, and Hongwei Zhang. Platoon control of connected vehicles from a networked control perspective: Literature review, component modeling, and controller synthesis. *IEEE Transactions on Vehicular Technology*, 2017.
- [111] Yang Zhu, Jun Wu, and Hongye Su. V2V-based cooperative control of uncertain, disturbed and constrained nonlinear CAVs platoon. *IEEE Transactions on Intelligent Transportation Systems*, 23(3):1796–1806, 2022.

- 
- [112] Yonggui Liu, Bugong Xu, and Yuehua Ding. Convergence analysis of cooperative braking control for interconnected vehicle systems. *IEEE Transactions on Intelligent Transportation Systems*, 18(7):1894–1906, 2016.
- [113] Hossein Chehardoli and Ali Ghasemi. Adaptive centralized/decentralized control and identification of 1-D heterogeneous vehicular platoons based on constant time headway policy. *IEEE Transactions on Intelligent Transportation Systems*, 19(10):3376–3386, 2018.
- [114] Rajesh Rajamani. *Vehicle dynamics and control*. Springer Science & Business Media, 2011.
- [115] Xiang-Gui Guo, Jian-Liang Wang, Fang Liao, and Rodney Swee Huat Teo. Cnn-based distributed adaptive control for vehicle-following platoon with input saturation. *IEEE Transactions on Intelligent Transportation Systems*, 19(10):3121–3132, 2018.
- [116] Yang Zheng, Shengbo Eben Li, Keqiang Li, and Wei Ren. Platooning of connected vehicles with undirected topologies: Robustness analysis and distributed h-infinity controller synthesis. *IEEE Transactions on Intelligent Transportation Systems*, 19(5):1353–1364, 2018.
- [117] Ali Ghasemi, Reza Kazemi, and Shahram Azadi. Stable decentralized control of a platoon of vehicles with heterogeneous information feedback. *IEEE Transactions on Vehicular Technology*, 62(9):4299–4308, 2013.
- [118] Jeroen Ploeg, Nathan van de Wouw, and Henk Nijmeijer. Lp string stability of cascaded systems: Application to vehicle platooning. *IEEE Transactions on Control Systems Technology*, 22(2):786–793, 2014.
- [119] Yang Zheng, Shengbo Eben Li, Keqiang Li, and Wei Ren. Platooning of connected vehicles with undirected topologies: Robustness analysis and distributed h-infinity controller synthesis. *IEEE Transactions on Intelligent Transportation Systems*, 19(5):1353–1364, 2017.
- [120] Shahab Sheikholeslam and Charles A Desoer. Longitudinal control of a platoon of vehicles with no communication of lead vehicle information: A system level study. *IEEE Transactions on vehicular technology*, 42(4):546–554, 1993.
- [121] Ge Guo and Wei Yue. Autonomous platoon control allowing range-limited sensors. *IEEE Transactions on vehicular technology*, 61(7):2901–2912, 2012.
- [122] Amir Firooznia, Jeroen Ploeg, Nathan Van De Wouw, and Hans Zwart. Co-design of controller and communication topology for vehicular platooning. *IEEE transactions on intelligent transportation systems*, 18(10):2728–2739, 2017.
- [123] Erkan Kayacan, Herman Ramon, and Wouter Saeys. Robust trajectory tracking error model-based predictive control for unmanned ground vehicles. *IEEE/ASME Transactions on Mechatronics*, 21(2):806–814, 2015.
- [124] Woongsun Jeon, Ali Zemouche, and Rajesh Rajamani. Tracking of vehicle motion on highways and urban roads using a nonlinear observer. *IEEE/ASME transactions on mechatronics*, 24(2):644–655, 2019.

- 
- [125] Chang Liu, Fen Liu, Hongxia Rao, and Yong Xu. Trajectory tracking with constrained sensors and unreliable communication networks. *IEEE Access*, 7:184866–184874, 2019.
- [126] Ryan James Caverly and James Richard Forbes. LMI properties and applications in systems, stability, and control theory. *arXiv preprint arXiv:1903.08599*, 2019.
- [127] Vladimir Andreevich Yakubovich. The solution of some matrix inequalities encountered in automatic control theory. In *Doklady Akademii Nauk*, volume 143, pages 1304–1307. Russian Academy of Sciences, 1962.
- [128] Vladimir Andreevich Yakubovich. Solution of certain matrix inequalities encountered in non-linear control theory. In *Doklady Akademii Nauk*, volume 156, pages 278–281. Russian Academy of Sciences, 1964.
- [129] Ryan James Caverly and James Richard Forbes. Lmi properties and applications in systems, stability, and control theory. *arXiv preprint arXiv:1903.08599*, 2019.



---

# Introduction

*“Dans le domaine de l’ingénierie des systèmes de contrôle, les observateurs non linéaires se présentent comme des outils fiables, révélant les dynamiques complexes et non linéaires des systèmes pour améliorer les stratégies d’estimation d’état.”*

---

## Contexte et Portée de la recherche

L’estimation de l’état des systèmes dynamiques est un sujet de recherche essentiel dans le domaine des systèmes de contrôle. La connaissance de ces informations en temps réel des systèmes obtenue à partir de l’estimation de l’état est utilisée pour des tâches liées à ces systèmes telles que la surveillance, le contrôle et la prise de décision. Ces outils jouent donc un rôle essentiel dans diverses applications, telles que l’auto-synchronisation dans les systèmes multi-agents [1] ; les cyber-attaques et la détection des défauts dans le peloton de véhicules autonomes connectés [2], et ainsi de suite. Dans la littérature, de nombreuses méthodologies ont été développées pour l’estimation de l’état des systèmes linéaires, par exemple les filtres de Kalman [3], l’observateur de Luenberger [4]. Cependant, la conception de tels outils pour les systèmes non linéaires reste une tâche difficile. Cela est dû à la structure complexe des non-linéarités du système. Plusieurs approches ont été proposées dans la littérature pour reconstruire les états des systèmes non linéaires, par exemple le filtre de Kalman étendu [5], le filtre de Kalman non accentué [6], les observateurs à grand-gain [7], les observateurs à mode glissant [8], etc. En plus de ces techniques, les observateurs non linéaires basés sur les LMIs ont récemment fait l’objet d’une grande attention dans le domaine du contrôle [9–12]. En dépit des diverses stratégies efficaces de conception d’observateurs basées sur le LMI présentées dans la littérature contemporaine, il existe un potentiel d’amélioration des approches basées sur le LMI. Ces possibilités d’amélioration des méthodologies basées sur le LMI ont incité l’auteur à mener ses travaux de recherche dans ce domaine. Les sections suivantes illustrent l’objectif principal et le cadre détaillé de cette thèse.

## Objectif

Comme indiqué précédemment, l’approche de la conception d’observateurs non linéaires basée sur le LMI est devenue l’un des sujets de recherche les plus récents dans le domaine de l’ingénierie des systèmes de contrôle. Plusieurs techniques d’observation basées sur les LMI sont présentées dans [12–17]. Certaines de ces approches reposent sur le lemme de la procédure-S [18], les équations de Riccati [19] et l’inégalité de Young [20]. Toutes ces méthodes fournissent

une condition LMI améliorée par rapport aux autres approches. Récemment, certaines conditions LMI améliorées ont été présentées dans [14, 15, 21, 22]. Bien que toutes ces méthodes LMI soient améliorées et moins conservatrices que les méthodes existantes, elles peuvent encore être améliorées. Parmi toutes les méthodologies susmentionnées, l'approche proposée dans [14] est basée sur la propriété de Lipschitz reformulée des non-linéarités et sur une variante réemment définie de l'inégalité de Young. L'un des éléments fondamentaux de cette méthode est l'introduction d'un multiplicateur matriciel bloc-diagonal inspiré de [12, 13]. La question qui se pose ici est de savoir pourquoi une structure matricielle bloc-diagonale seulement. Cette question incite les auteurs à proposer une nouvelle condition LMI basée sur un multiplicateur de matrice plus généralisé qu'une matrice diagonale. L'inclusion d'un tel multiplicateur de matrice dans les LMI peut être bénéfique en termes d'amélioration des LMI existantes. En résumé, cette thèse se concentre principalement sur le développement de nouvelles conditions LMI qui ont plus de degrés de liberté et une région de faisabilité plus grande que celles présentées dans [13, 14, 22]. Les objectifs du travail de recherche présenté dans cette thèse sont décrits comme suit:

- **Développer un nouvel observateur basé sur les LMIs:** L'objectif principal est de proposer une approche de conception d'observateurs pour la classe des systèmes Lipschitz globalement non linéaires en utilisant les nouvelles conditions LMI, qui sont moins conservatrices que l'approche existante et offrent plus d'améliorations du point de vue de la faisabilité.
- **Utiliser l'approche LMI proposée pour plusieurs applications de contrôle:** Dans la deuxième partie de cette thèse, les auteurs visent à mettre en œuvre l'approche LMI basée sur le multiplicateur de matrice développée précédemment pour estimer les états des systèmes non linéaires affectés par des perturbations. En outre, la thèse se concentre sur la conception d'observateurs non linéaires adaptés aux systèmes Lipschitz non globaux. Plus tard, cette thèse approfondira le sujet de l'approche de stabilisation basée sur l'observateur pour les systèmes non linéaires utilisant les LMI.
- **Appliquer les techniques d'observation développées à la technologie des véhicules autonomes:** Le but de la dernière partie de cette thèse est de déployer les observateurs non linéaires proposés dans diverses tâches de véhicules autonomes.

Dans la partie suivante, nous présentons la structure de cette thèse.

## L'organisation de la thèse

Le plan détaillé de la thèse est décrit de la manière suivante:

- I) Une vue d'ensemble de certaines notions importantes nécessaires à la conception d'un observateur et de plusieurs techniques d'observation existantes est présentée dans **Chapitre 1**. Pour faciliter la compréhension, **Chapitre 1** est structuré comme suit: La section 1.2 est consacrée à la présentation de certains concepts fondamentaux qui sont essentiels pour la conception d'observateurs non linéaires. La section 1.3 constitue un résumé des diverses méthodologies de conception d'observateurs non linéaires existantes. Enfin, la section 1.4 contient quelques conclusions.
- II) **Chapitre 2** est dédié à l'introduction de nouvelles conditions LMI basées sur le multiplicateur de matrice qui sont moins conservatrices que les conditions existantes. Le chapitre est organisé comme suit: Un aperçu de l'observateur basé sur le LMI et la motivation de la méthodologie proposée sont discutés dans la section 2.2. Les conditions préalables requises

pour les nouvelles méthodes de conception d'observateurs sont illustrées dans la section 2.3. Une nouvelle approche LMI basée sur le multiplicateur de matrice est présentée dans la section 2.4. En outre, la section 2.5 contient une approche LMI améliorée basée sur le multiplicateur de matrice. La section 2.5 comprend la validation des techniques proposées à l'aide d'exemples numériques. La section 2.7 présente quelques conclusions.

- III) Dans la suite de cette thèse, **Chapitre 3** englobe la mise en œuvre des nouvelles techniques d'observateur basées sur LMI proposées pour plusieurs tâches, telles que l'estimation de l'état du système non linéaire affecté par des perturbations et la conception d'observateur pour le système non globalement Lipschitz. La configuration de ce chapitre est illustrée comme suit: La section 3.2 englobe la conception d'un observateur non linéaire  $\mathcal{H}_\infty$ . Le développement de la condition LMI basée sur l'inégalité standard de Young est illustré dans la sous-section 3.2.1, tandis que la sous-section 3.2.2 inclut la formulation de l'approche LMI basée sur LPV. L'extension de la nouvelle approche LMI basée sur le multiplicateur de matrice proposée pour le système non linéaire avec des non-linéarités locales de Lipschitz est présentée dans la section 3.3. En outre, la section 3.4 met l'accent sur l'effet des différentes structures des multiplicateurs de matrice sur les LMIs proposée. L'amélioration des conditions LMI développées et la validation de l'observateur proposé sont démontrées dans la section 3.5. Quelques conclusions sont tirées dans la section 3.6.
- IV) En outre, dans **Chapitre 4**, une stratégie de stabilisation basée sur l'observateur pour les systèmes non linéaires est établie en utilisant notre approche LMI basée sur le multiplicateur de matrice récemment développée dans notre travail. Nous avons organisé ce chapitre de la manière suivante: La section 4.2 est consacrée à la formulation de l'énoncé du problème. Pour le contrôleur basé sur l'observateur proposé, une nouvelle condition LMI est formulée dans la Section 4.3 en utilisant le critère  $\mathcal{H}_\infty$ . En outre, la propriété de l'ISS est utilisée dans la section 4.4 pour dériver un critère LMI. La performance des approches proposées est validée dans la section 4.5 à l'aide d'exemples numériques. Enfin, la section 4.6 présente quelques remarques finales.
- V) Plus tard, le déploiement de l'observateur proposé dans le domaine des véhicules autonomes est présenté dans **Chapitre 5**. Le plan de ce chapitre est illustré comme suit: La section 5.2 est consacrée à la mise en œuvre des observateurs proposés au chapitre 3 sur le modèle de véhicule longitudinal. Le déploiement de l'observateur basé sur l'ISS du chapitre 3 aux fins de l'estimation de l'angle de glissement du modèle de véhicule non linéaire est illustré à la section 5.3. En outre, La section 5.4 englobe le développement de l'observateur non linéaire basé sur la transformation de coordonnées pour un modèle de suivi de véhicule. Enfin, certaines conclusions sont présentées dans la section 5.5.

Dans le segment suivant, les auteurs illustrent les contributions du travail de recherche présenté dans cette thèse.

## La contribution de la thèse

La contribution globale de cette thèse est listée comme suit:

- A. **L'établissement de conditions LMI nouvelles et enrichies:** L'un des résultats notables de cette thèse est le développement de nouvelles conditions LMI basées sur un multiplicateur de matrice. Ces conditions sont formulées en intégrant le multiplicateur matriciel récemment

défini à une approche LPV bien établie et à une propriété de Lipschitz reformulée récemment introduite. En raison de l'utilisation judicieuse du multiplicateur de matrice récemment proposé, les conditions LMI obtenues contiennent des nombres supplémentaires de variables de décision que les approches LMI existantes [12–14]. Les variables de décision supplémentaires ont élargi la région de faisabilité de la LMI, améliorant ainsi ces conditions du point de vue de la faisabilité.

B. **L'applicabilité de la méthode LMI proposée:** L'approche LMI développée est appliquée à diverses tâches de contrôle, comme décrit ci-dessous:

- 1) L'observateur basé sur le LMI est utilisé pour concevoir un observateur  $\mathcal{H}_\infty$  amélioré pour estimer l'état des systèmes non linéaires affectés par des perturbations externes et du bruit à la fois dans leur dynamique et leurs sorties. Les conditions LMI établies permettent une meilleure atténuation du bruit que les méthodes existantes de [12–14], comme le montrent les exemples numériques.
- 2) En outre, pour les systèmes non linéaires Lipschitz non globaux affectés par des perturbations, l'observateur proposé dans [23] est incorporé à l'approche LMI récemment développée pour l'estimation de l'état. Les conditions LMI obtenues sont formulées en utilisant la notion d'ISS. L'utilisation de cette méthode permet d'assouplir l'hypothèse principale de non-linéarités globales de Lipschitz requise pour les problèmes de conception d'observateurs non linéaires.
- 3) Par la suite, la nouvelle condition LMI est conçue pour la stabilisation basée sur l'observateur d'une classe de systèmes non linéaires en intégrant des multiplicateurs de matrice récemment définis, la propriété de Lipschitz reformulée et les inégalités de Young. En raison du déploiement délibéré de ces outils mathématiques dans la formulation des LMI, les conditions résultantes sont améliorées et moins conservatrices que celles présentées dans [24–26].

C. **Le déploiement des observateurs proposés dans plusieurs applications de véhicules autonomes:** La méthodologie de l'observateur basé sur le LMI formulée est déployée pour diverses tâches de véhicules autonomes, y compris l'estimation de l'état longitudinal, le suivi du véhicule et l'estimation de l'angle de glissement. Elle permet de valider les performances de l'observateur proposé dans des applications pratiques.

---

# Conclusion et perspectives

*“De chaque fin, il y a un nouveau commencement qui attend d’être écrit.”*

---

*Unknown*

Cette thèse porte sur la conception d’observateurs non linéaires à l’aide de techniques de relaxation LMI. Dans la première partie, les auteurs ont fourni un bref résumé des notions fondamentales utilisées dans le développement d’observateurs non linéaires et de plusieurs méthodologies de conception d’observateurs. En particulier, certains concepts importants d’observabilité et de stabilité sont illustrés. En outre, certains outils mathématiques nécessaires à la conception de l’observateur ont été rappelés. Tous ces éléments fondamentaux ont été utilisés dans les parties suivantes de cette thèse pour la formulation des nouvelles conditions LMI. En outre, quelques nouveaux observateurs basés sur le LMI ont été proposés dans cette thèse. En utilisant judicieusement les multiplicateurs de matrice nouvellement définis, à savoir (2.82) et (2.83) ; en appliquant les inégalités de Young (A.9) et (A.10) ; et en incorporant la propriété de Lipschitz reformulée, comme indiqué dans la Définition 3 de l’Annexe A.1, de nouvelles conditions LMI ont été établies. Ces LMIs obtenus offrent une meilleure faisabilité pour des constantes de Lipschitz plus grandes que celles présentées dans la littérature, ce qui est démontré par un exemple numérique. L’inclusion d’un multiplicateur de matrice généralisé dans la formulation LMI introduit des variables de décision supplémentaires dans ces conditions, augmentant les degrés de liberté et améliorant par conséquent leur faisabilité. Les performances des observateurs basés sur le LMI développés sont validées à l’aide d’exemples numériques.

Par la suite, les conditions LMI établies sont déployées dans le cas de la conception d’observateurs non linéaires  $\mathcal{H}_\infty$ . Les LMI obtenues sont moins conservatrices que celles présentées dans [12–14], et ceci est démontré par des exemples numériques. En outre, cette technique est associée à l’observateur de Luenberger basé sur la projection de Hilbert récemment proposé dans [23] pour l’estimation de l’état du système Lipschitz non global affecté par des perturbations. Plus loin dans cette thèse, nous avons conçu la stratégie de stabilisation basée sur l’observateur pour les systèmes non linéaires à l’aide des LMI susmentionnés.

Dans la dernière partie de la thèse, l’applicabilité de l’observateur basé sur le LMI établi dans le domaine des véhicules autonomes est démontrée. Nous avons déploré l’observateur basé sur le LMI proposé au Chapitre 3 pour estimer l’état longitudinal d’un modèle de véhicule autonome non linéaire du troisième ordre. Les auteurs ont utilisé l’observateur non linéaire basé sur l’ISS nouvellement dérivé du chapitre 3 pour estimer l’angle de glissement des véhicules autonomes. Par la suite, le problème de suivi de véhicule est résolu en intégrant la méthodologie de l’observateur basé sur le LMI proposé dans le Chapitre 2. Ces applications permettent donc

de vérifier et d'analyser les performances des observateurs basés sur le LMI proposés.

## Les perspectives de la recherche

L'une des principales contributions de cette thèse est le développement des méthodes d'observateur basé sur le LMI pour la conception d'observateurs non linéaires. Considérant l'importance des systèmes non linéaires et des LMI dans les systèmes de contrôle, les approches proposées ouvrent de nouvelles possibilités de recherche dans divers aspects de l'ingénierie du contrôle. Du point de vue de l'auteur, les tâches futures potentielles liées aux nouvelles techniques LMI sont énumérées ci-dessous :

1. **Construction d'observateurs non linéaires pour les systèmes à temps discret:** L'utilisation des propriétés de Lipschitz reformulées des non-linéarités, des inégalités de Young et des multiplicateurs de matrice proposés dans l'établissement d'observateurs non linéaires pour les systèmes à temps continu a permis d'améliorer les conditions LMI. Nous avons l'intention d'appliquer une approche similaire pour développer des observateurs pour les systèmes de Lipschitz non linéaires à temps discret, afin de dériver des conditions LMI moins conservatrices que celles existantes. Par la suite, nous utiliserons l'observateur proposé pour accomplir des tâches telles que le suivi ou la stabilisation de la sortie dans ces systèmes.
2. **Exploiter les observateurs proposés pour les systèmes incertains.** : Dans les systèmes non linéaires complexes, l'incertitude présente dans la dynamique ou les sorties est l'un des obstacles critiques au cours du processus d'estimation de l'état. Par conséquent, les chercheurs en systèmes de contrôle ont montré un niveau d'intérêt notable dans ce domaine. Les auteurs ont l'intention d'utiliser l'approche LMI établie dans ce domaine, ce qui pourrait aboutir à des outils d'estimation d'état efficaces pour ces types de systèmes.
3. **Estimation des signaux exogènes, tels que les défauts, les perturbations et les cyber-attaques, présents dans les systèmes non linéaires.** : La progression continue des technologies modernes a rendu les systèmes complexes non linéaires plus vulnérables aux influences extérieures, telles que le bruit, les défaillances et les cyber-attaques. Outre la supervision de l'état de ces systèmes, il est essentiel de recueillir des données sur ces signaux exogènes afin d'atténuer le risque de défaillance du système. Les auteurs avaient prévu de résoudre ce problème en incorporant les méthodes existantes d'observateurs à entrées inconnues aux techniques LMI proposées.

En outre, comme les auteurs l'ont mentionné précédemment, l'approche LMI basée sur les multiplicateurs de matrice peut être déployée dans d'autres problèmes de contrôle.

