



**HAL**  
open science

# Contribution to functional observer design. Application to control and diagnosis

Carlos Ríos-Ruiz

► **To cite this version:**

Carlos Ríos-Ruiz. Contribution to functional observer design. Application to control and diagnosis. Automatic. Université de Lorraine; Tecnológico nacional de México - CENIDET, 2023. English. NNT : 2023LORR0252 . tel-04624813

**HAL Id: tel-04624813**

**<https://hal.univ-lorraine.fr/tel-04624813>**

Submitted on 25 Jun 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



**UNIVERSITÉ  
DE LORRAINE**

**BIBLIOTHÈQUES  
UNIVERSITAIRES**

## AVERTISSEMENT

Ce document est le fruit d'un long travail approuvé par le jury de soutenance et mis à disposition de l'ensemble de la communauté universitaire élargie.

Il est soumis à la propriété intellectuelle de l'auteur. Ceci implique une obligation de citation et de référencement lors de l'utilisation de ce document.

D'autre part, toute contrefaçon, plagiat, reproduction illicite encourt une poursuite pénale.

Contact bibliothèque : [ddoc-theses-contact@univ-lorraine.fr](mailto:ddoc-theses-contact@univ-lorraine.fr)  
*(Cette adresse ne permet pas de contacter les auteurs)*

## LIENS

Code de la Propriété Intellectuelle. articles L 122. 4

Code de la Propriété Intellectuelle. articles L 335.2- L 335.10

[http://www.cfcopies.com/V2/leg/leg\\_droi.php](http://www.cfcopies.com/V2/leg/leg_droi.php)

<http://www.culture.gouv.fr/culture/infos-pratiques/droits/protection.htm>

---

Département de formation doctorale en automatique  
UFR Sciences et Technologie

École doctorale IAEM Lorraine

# Contribution to Functional Observer Design. Application to Control and Diagnosis

## THESIS

Presented and publicly defended on November 24<sup>th</sup>, 2023  
to obtain the

**Doctor degree from the University of Lorraine**  
(Speciality : Automatic, Signal Processing, and Computer Science)  
by:

**Carlos RIOS RUIZ**

### Composition of the jury

<i>Thesis</i>	M. Mohamed DAROUACH	Professor. Université de Lorraine, CRAN. Longwy
<i>Director:</i>	M. Rachid OUTBIB	Professor, Université de Marseille
<i>President:</i>	M. Flor Lizeth TORRES	Professor, Universidad Autónoma de México. Mexico, Mexique
<i>Reporter:</i>	Mme. Gloria Lilia OSORIO GORDILLO	Professor, Centro Nacional de Investigación y Desarrollo Tecnológico CENIDET. Cuernavaca, Mexique.
<i>Reviewers:</i>	M. Carlos Manuel ASTORGA ZARAGOZA	Professor, Centro Nacional de Investigación y Desarrollo Tecnológico CENIDET. Cuernavaca, Mexique.
	M. Harouna SOULEY-ALI	Maître de conférences, Université de Lorraine, CRAN. Co-director of thesis. Longwy

## Agradecimientos

Quiero agradecer profundamente a mi familia, mis padres y hermana por su constante apoyo y sus palabras de aliento, sin ellos no hubiera logrado terminar este trabajo.

A mis amigos y compañeros, por todos los momentos que pasamos y por el apoyo que recibí de ellos. A Sarai Campechano, Susana Sainz y Víctor Rodríguez, sin su ayuda consejos y cariño este proyecto no hubiera sido posible.

Agradezco a mis asesores, Dr. Carlos Manuel Astorga Zaragoza, Dr. Mohamed Darouach, Dr. Harouna Souley-Ali y Dra. Gloria Lilia Osorio Gordillo. Gracias por sus todo el conocimiento, orientaciones, su infinita paciencia y sobretodo por su amistad. Gracias por formarme.

A mi comité revisor, Dr. Carlos Daniel García Beltrán, Dr. Víctor Manuel Alvarado Martínez y Dr. Juan Reyes Reyes por toda la orientación y los comentarios que ayudaron a enriquecer este trabajo.

Gracias al Consejo Nacional de Humanidades, Ciencias y Tecnologías (CONAH-CYT) por su apoyo. Al Tecnológico Nacional de México por darme tantos años de logros y satisfacción y a l'Université de Lorraine por abrirme las puertas y permitirme volar más allá de lo que hubiera imaginado.



# Contents

<b>List of figures</b>	<b>VI</b>
<b>Notation</b>	<b>VII</b>
<b>Abstract</b>	<b>IX</b>
<b>Résumé étendu</b>	<b>XI</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Background . . . . .	1
1.2 State of the art . . . . .	3
1.2.1 Descriptor systems . . . . .	3
1.2.2 Observers . . . . .	8
1.3 Objectives . . . . .	10
1.3.1 General objective . . . . .	10
1.3.2 Specific objectives . . . . .	11
1.4 Justification . . . . .	11
1.5 Originality . . . . .	11
1.6 Document layout . . . . .	12
<b>2 Theoretical Framework</b>	<b>13</b>
2.1 Descriptor systems . . . . .	14
2.1.1 Examples . . . . .	14
2.1.2 Descriptor systems properties . . . . .	18
2.2 LMI regions . . . . .	24
2.2.1 $\mathcal{D}$ -stability . . . . .	24
2.3 Takagi-Sugeno systems . . . . .	25
2.4 Functional observers . . . . .	26
2.5 Conclusion . . . . .	27
<b>3 Functional observers for LTI systems</b>	<b>29</b>
3.1 Functional observer for descriptor systems . . . . .	30
3.1.1 Class of systems considered . . . . .	30
3.1.2 Finite-Time Generalized Functional Observer Design . . . . .	32

3.1.3	GDO design for LTI systems . . . . .	40
3.2	Functional observer for descriptor systems with unknown input . . . . .	49
3.2.1	Problem statement . . . . .	50
3.2.2	Observer parameterization . . . . .	52
3.2.3	Observer design . . . . .	55
3.3	Functional observer for T-S descriptor systems . . . . .	57
3.3.1	Problem statement . . . . .	57
3.3.2	Observer parameterization . . . . .	59
3.3.3	Functional observer design . . . . .	61
3.3.4	Particular case: Proportional Functional observer . . . . .	62
3.3.5	Mathematical model . . . . .	63
3.3.6	Results . . . . .	64
3.3.7	Application to state and fault estimation for LTI systems . . . . .	67
3.4	Conclusions . . . . .	70
<b>4</b>	<b>Application of functional observers</b>	<b>73</b>
4.1	Functional observer-based control . . . . .	74
4.1.1	Problem statement . . . . .	74
4.1.2	Controller design . . . . .	76
4.1.3	Numerical example . . . . .	78
4.2	Sensor fault-tolerant control . . . . .	80
4.2.1	Problem statement . . . . .	80
4.2.2	Numerical example . . . . .	81
4.3	Actuator fault-tolerant control . . . . .	83
4.3.1	Problem statement . . . . .	85
4.3.2	Numerical example . . . . .	85
4.4	Simultaneous state estimation and fault-tolerant control . . . . .	88
4.4.1	Problem statement . . . . .	88
4.4.2	Numerical example . . . . .	89
4.5	Conclusions . . . . .	92
<b>5</b>	<b>Conclusions</b>	<b>95</b>
5.1	Contributions . . . . .	96
5.2	Products . . . . .	97
5.3	Discussion . . . . .	97
5.4	Limitations . . . . .	98
5.5	Future works . . . . .	99
	<b>Bibliography</b>	<b>102</b>

# List of Figures

1.1	Classical observer-based control scheme with $u(t) = K\hat{x}(t) + u_d(t)$ . . . . .	2
1.2	Interconnected water tanks used in [Araujo et al., 2012] . . . . .	4
1.3	Human body stance represented as a double pendulum . . . . .	5
2.1	Electrical - Mechanical system . . . . .	15
2.2	Single link robot arm . . . . .	16
2.3	$\mathcal{D}$ -stability region . . . . .	25
3.1	Block diagram of the control system . . . . .	40
3.2	States of the LTI descriptor system . . . . .	41
3.3	Estimated control law for the LTI descriptor system . . . . .	41
3.4	Estimated states . . . . .	49
3.5	Single link robot arm . . . . .	63
3.6	Results of the simulation. . . . .	66
3.7	Uncertainty variation . . . . .	69
3.8	Block diagram of the functional observer . . . . .	69
3.9	Function estimation . . . . .	70
3.10	Fault estimation . . . . .	71
4.1	Block diagram for the observer-based control system . . . . .	75
4.2	States of the system . . . . .	79
4.3	Estimated control law . . . . .	79
4.4	Block Diagram for sensor a FTC system . . . . .	80
4.5	Sensor fault . . . . .	82
4.6	States of the system . . . . .	82
4.7	Estimated control law . . . . .	83
4.8	States of the system . . . . .	84
4.9	Fault-tolerant control scheme . . . . .	84
4.10	Faults in the system $f(t)$ . . . . .	87
4.11	$u_r(t)$ and $\hat{u}(t)$ . . . . .	87
4.12	Comparison between $x(t)$ and $x_d(t)$ . . . . .	88
4.13	Estimated $\hat{f}(t)$ and its comparison with $f(t)$ . . . . .	90
4.14	Estimated control law . . . . .	91
4.15	States of the system . . . . .	92



# Notation

Symbol	Description	Units
$m$	Mass	$Kg$
$F$	Force	$N$
$J$	Moment of inertia	$Kgm^2$
$v$	Linear velocity	$m/s$
$g$	Gravity acceleration	$m/s^2$
$V$	Voltage	$V$
$k$	Spring coefficient	$N/m$
$R$	Resistor value	$\Omega$
$L$	Inductance	$H$
$C$	Capacitance	$F$
$K$	Controller gains	-

## Greek letters

Symbol	Description	Units
$\theta$	Angle of rotation	$rad$
$\omega$	Angular velocity	$rad/s$
$\tau$	Torque	$Nm$

## Acronyms

Symbol	Description
LTI	Linear Time-Invariant
GDO	Generalized Dynamical Observer
LMI	Linear Matrix Inequality
T-S	Takagi-Sugeno



# Abstract

The importance of fault diagnosis and fault-tolerant control systems has been rising. The complex systems used in our daily life require high safety and reliability features, thus fault diagnosis and fault-tolerant systems have an increasing importance in control research.

The reconstruction of the system state has great importance in control theory. Numerous feedback controller require the measurement of all the state for the stabilization of the systems. However, for descriptor systems the conditions for the observer existence are different from standard systems.

This thesis presents a comprehensive study on observer-based control for descriptor systems, with a focus on the application of functional observers and fault-tolerant control methodologies. The research explores the theoretical foundations, design methodologies, and practical implications of functional observers in accurately estimating the states of descriptor systems. The thesis investigates the synthesis of fault-tolerant control strategies to ensure system stability and performance in the presence of sensor and actuator faults. Additionally, the integration of fault estimation techniques within the control framework is explored to achieve simultaneous fault-tolerant control and fault diagnosis capabilities. Theoretical advancements are complemented by extensive simulation-based evaluations, demonstrating the efficacy of the proposed methodologies in various scenarios. Furthermore, the limitations and future research directions are discussed, highlighting the need to address challenges such as actuator and sensor faults, handling nonlinear systems, considering the effects of noise in output measurements, and practical implementation considerations. This research contributes to the advancement of observer-based control for descriptor systems, offering valuable insights and methodologies that can enhance system reliability, performance, and fault tolerance in real-world applications.



# Résumé étendu

L'importance du diagnostic de défauts et des systèmes de contrôle tolérants aux défauts a augmenté les dernières années. Les systèmes complexes utilisés dans notre vie quotidienne exigent des caractéristiques de sécurité et de fiabilité élevées, c'est pourquoi le diagnostic de défauts et les systèmes tolérants aux défauts ont une importance croissante dans la recherche en contrôle.

La reconstruction de l'état du système revêt une grande importance dans la théorie du contrôle. De nombreux contrôleurs en boucle fermée nécessitent la mesure de l'ensemble de l'état pour la stabilisation des systèmes. Cependant, pour les systèmes descripteurs, les conditions d'existence de l'observateur diffèrent des systèmes standards.

Cette thèse présente une étude complète sur le contrôle basé sur l'observateur pour les systèmes descripteurs, en mettant l'accent sur l'application des observateurs fonctionnels et des méthodologies de contrôle tolérant aux défauts. Les travaux de recherche explorent les bases théoriques, les méthodologies de conception et les implications pratiques des observateurs fonctionnels pour estimer avec précision les états des systèmes descripteurs. La thèse examine la synthèse de stratégies de contrôle tolérantes aux défauts pour garantir la stabilité et les performances du système en présence de défauts de capteurs et d'actionneurs. De plus, l'intégration de techniques d'estimation de défaut dans le cadre de contrôle est explorée pour obtenir simultanément des capacités de contrôle tolérant aux défauts et de diagnostic de défaut. Les avancées théoriques sont complétées par des évaluations approfondies basées sur des simulations, démontrant l'efficacité des méthodologies proposées dans divers scénarios. De plus, les limitations et les orientations de recherche futures sont discutées, mettant en lumière la nécessité de relever des défis tels que les défauts d'actionneurs et de capteurs, la prise en compte des systèmes non linéaires, la considération des effets du bruit dans les mesures de sortie, et les considérations de mise en œuvre pratique. Cette recherche contribue à l'avancement du contrôle basé sur l'observateur pour les systèmes descripteurs, offrant des aperçus et des méthodologies précieuses qui peuvent améliorer la fiabilité, les performances et la tolérance aux défauts des systèmes dans des applications du monde réel.

On commence avec le chapitre appelé Introduction. Ce chapitre sert d'introduction à la thèse, présentant les motivations, les avancées récentes et les objectifs généraux de la recherche. Il effectue une revue complète, posant les bases pour les analyses ultérieures. L'état de l'art des systèmes descripteurs et des observateurs est examiné, détaillant des objectifs clairs. La justification de la recherche est soulignée, mettant en avant sa pertinence. Le chapitre se conclut en résumant les contributions originales et en fournissant un aperçu de la structure du

document.

La section sur le contexte explore l'importance de l'estimation d'état dans les systèmes de contrôle depuis l'introduction du concept par Luenberger en 1966. Elle illustre la nécessité d'estimer des états non mesurables et présente des schémas de contrôle basés sur des observateurs. Les observateurs sont discutés en tant qu'outils en théorie de contrôle, capables de gérer diverses applications, y compris le diagnostic de panne et le contrôle tolérant aux pannes. Le texte souligne que ce domaine offre des opportunités d'exploration, avec un potentiel pour de nouvelles applications et développements d'observateurs.

De plus, la section aborde les systèmes linéaires descripteurs, largement étudiés pour représenter des systèmes dynamiques dans divers domaines. Ces systèmes offrent une manière naturelle de décrire le comportement des systèmes dynamiques en mécanique, processus chimiques, génie électrique et économie. Les systèmes linéaires descripteurs sont mis en avant comme des outils puissants pour résoudre des problèmes de contrôle dans des systèmes linéaires standards avec des perturbations ou des défauts inconnus.

On aborde l'état de l'art des observateurs fonctionnels et du contrôle pour les systèmes descripteurs, mettant en avant la diversité des recherches et des avancées dans ce domaine. Des progrès significatifs ont été réalisés dans la conception de modèles d'observateurs robustes, de stratégies de contrôle tolérantes aux pannes et de techniques d'intégration pour gérer les incertitudes dans des applications du monde réel. La discussion englobe diverses approches visant à améliorer les performances, la stabilité et la tolérance aux pannes des systèmes descripteurs, en tenant compte de la complexité des dynamiques, des non-linéarités et des défis liés à la mise en œuvre pratique.

La section sur les systèmes descripteurs explique leur représentation, mettant en évidence leur capacité à modéliser des systèmes avec des équations dynamiques et algébriques. Des exemples incluent des applications dans les systèmes hydrauliques et les systèmes biomécaniques, illustrant la polyvalence des systèmes descripteurs. Le texte passe en revue les aspects historiques et théoriques des systèmes descripteurs, couvrant la stabilité, la stabilisation et la représentation des pannes.

La section suivante sur les observateurs donne un aperçu de leur importance historique, en mettant l'accent sur leur rôle en tant que capteurs virtuels lorsque certaines parties de l'état ne sont pas mesurables. Il aborde la sensibilité des observateurs proportionnels classiques aux incertitudes et aux perturbations, introduisant les observateurs proportionnels-intégraux comme solution. Diverses conceptions et applications d'observateurs fonctionnels sont explorées, y compris leur utilisation pour stabiliser des systèmes instables et estimer les états dans des systèmes descripteurs. Le texte mentionne également des études sur l'estimation d'état pour les systèmes descripteurs, englobant différentes structures et méthodologies d'observateurs.

La partie conclusive résume les domaines de recherche en cours dans l'état de l'art des systèmes LTI descripteurs. Elle souligne des aspects clés, tels que la modélisation et la représentation, l'analyse de la stabilité et du contrôle, la conception d'observateurs, la conception et l'optimisation du contrôle, ainsi que les applications concrètes. Le paysage dynamique des progrès en cours et des défis est reconnu, avec l'identification d'opportunités pour des investigations ultérieures. La vue d'ensemble complète sert de base pour l'exploration ultérieure de nouvelles méthodologies et avancées dans le domaine. Enfin, le texte souligne que l'état de l'art des observateurs et des

observateurs fonctionnels reste un domaine dynamique avec des études en cours axées sur des conceptions avancées, une analyse de la robustesse, le diagnostic des pannes et l'intégration des observateurs avec des stratégies de contrôle dans diverses applications.

Le chapitre 2, nommé "*Theoretical Framework*" sert de base à la recherche menée dans cette thèse sur les observateurs fonctionnels et le contrôle des systèmes descripteurs. Il offre un aperçu complet des bases théoriques et des concepts clés essentiels à la compréhension des chapitres suivants. En explorant les principes fondamentaux et les outils mathématiques utilisés dans l'analyse et la conception des observateurs fonctionnels et des stratégies de contrôle, ce chapitre établit un cadre théorique solide. Grâce à ce cadre théorique complet, ce chapitre prépare le terrain pour les chapitres suivants afin d'approfondir des aspects plus spécifiques des observateurs fonctionnels et du contrôle des systèmes descripteurs, fournissant les bases théoriques nécessaires pour faire progresser la compréhension et l'application de ces techniques dans des scénarios pratiques.

Il commence par un examen détaillé des systèmes descripteurs, démêlant leurs subtilités et explorant des exemples qui illustrent leur pertinence dans des applications pratiques. Les propriétés intrinsèques aux systèmes descripteurs sont ensuite scrutées, mettant en lumière les caractéristiques clés qui serviront de base aux discussions ultérieures. Ensuite, l'exploration s'étend aux régions des inégalités matricielles linéaires (LMI), en mettant particulièrement l'accent sur la D-stabilité, offrant une perspective mathématique permettant d'évaluer la stabilité du système.

Le chapitre se penche ensuite sur les systèmes Takagi-Sugeno, explorant leurs caractéristiques uniques et leur applicabilité à la gestion de systèmes complexes et non linéaires. Enfin, le cadre théorique englobe les observateurs fonctionnels, élucidant leur rôle dans l'observation et le contrôle du système. En établissant une base théorique solide, ce chapitre propulse les sections suivantes vers des aspects plus nuancés des observateurs fonctionnels et du contrôle des systèmes descripteurs. Cette base est essentielle pour faire progresser la compréhension et l'application pratique de ces techniques dans des scénarios du monde réel, fournissant une boîte à outils complète pour les chercheurs et les praticiens.

Le chapitre 2 aborde aussi l'importance de la reconstruction de l'état en théorie du contrôle et introduit le concept d'observateurs, à partir des travaux de D. Luenberger en 1966. Il détaille l'observateur de Luenberger pour les systèmes linéaires invariants dans le temps (LTI) et souligne la nécessité de la convergence de l'erreur d'estimation vers zéro en utilisant la stabilité de Lyapunov. Le texte introduit ensuite les observateurs fonctionnels, mettant en avant leur capacité à estimer une fonction du vecteur d'état sans nécessairement déterminer l'état complet. Les avantages des observateurs fonctionnels par rapport aux observateurs d'état d'ordre complet sont présentés, mettant l'accent sur une complexité réduite et l'estimation d'états spécifiques et de signaux de contrôle.

Le concept de convergence en temps fini est introduit, notant que les observateurs en temps continu ont une convergence asymptotique, tandis que les observateurs en temps discret peuvent converger en un nombre fini d'étapes. Un lemme est fourni pour une utilisation ultérieure, et la conclusion du chapitre reflète sur les aspects fondamentaux couverts, préparant le terrain pour une exploration plus approfondie dans les chapitres suivants.

Le chapitre 3, nommé “*Functional observers for LTI systems*” aborde l’importance de la reconstruction de l’état dans les applications pratiques du contrôle, telles que le diagnostic de défauts. Il souligne la difficulté d’obtenir le vecteur d’état complet en raison de coûts de mesure exorbitants ou de contraintes physiques. Dans de tels scénarios, un estimateur s’avère inestimable pour générer une estimation  $\hat{x}(t)$  du vecteur d’état  $x(t)$ . Avec la complexité croissante des systèmes contrôlés, l’accent est mis sur la sécurité et la fiabilité dans les schémas de détection de défaut et d’estimation.

Les défauts, définis comme des altérations imprévisibles du comportement du système entraînant une dégradation des performances, sont devenus un point central dans la conception des systèmes de contrôle. Le développement d’un contrôle tolérant aux défauts efficace repose sur l’obtention d’estimations précises des variables d’état et des défauts. La convergence en temps fini a émergé comme un domaine d’étude captivant, trouvant des applications dans divers domaines tels que les systèmes de communication, les systèmes mécaniques et les systèmes non linéaires. La convergence en temps fini revêt une importance particulière dans l’estimation de défauts en raison du besoin prédominant de contraintes de réponse rapide dans les systèmes de contrôle pratiques.

Dans ce chapitre, un estimateur convergent en temps fini est introduit en tirant parti du cadre des observateurs fonctionnels généralisés. En capitalisant sur la structure inhérente des observateurs fonctionnels, un observateur d’ordre réduit est présenté pour fournir simultanément des estimations en temps fini de l’état du système et des éventuelles défauts. Cette approche novatrice améliore l’efficacité et la précision de l’estimation de l’état et des défauts dans une période de temps finie définie.

La principale contribution du travail consiste à proposer un observateur fonctionnel asymptotique généralisé qui étend les résultats présentés dans [Darouach, 2012]. Ensuite, une combinaison de deux observateurs fonctionnels généralisés est utilisée pour concevoir un observateur fonctionnel généralisé en temps fini. Les résultats obtenus sont ensuite appliqués pour estimer une fonction linéaire de l’état en temps fini.

Les résultats sont alors étendus pour concevoir des observateurs fonctionnels pour des systèmes linéaires qui présentent des entrées inconnues et pour des systèmes non linéaires avec la représentation Takagi-Sugeno, pour l’estimation d’une fonction de l’état.

En conclusion, ce chapitre présente une synthèse des observateurs fonctionnels généralisés et de leur application à la détection de défaut, mettant en avant l’importance de la convergence en temps fini. Les résultats obtenus démontrent la pertinence de cette approche dans le domaine du contrôle des systèmes complexes et ouvrent la voie à des applications pratiques dans des contextes diversifiés.

Le chapitre 4, nommé “*Application of functional observers*”, constitue une exploration approfondie du contrôle basé sur l’observateur dans le contexte de l’approche généralisée des observateurs fonctionnels. Le contrôle basé sur l’observateur, un pilier de l’ingénierie moderne, joue un rôle vital dans la réalisation d’une régulation précise et de la manipulation de processus dynamiques. En estimant avec précision les états internes d’un système, les ingénieurs peuvent élaborer des stratégies de contrôle robustes, réactives et polyvalentes. L’objectif de ce

chapitre est de fournir une compréhension approfondie du contrôle basé sur l'observateur, de ses principes fondamentaux, de ses méthodologies de conception et de ses applications pratiques.

À travers l'exploration des observateurs fonctionnels, ce chapitre vise à fournir une compréhension complète de leurs méthodologies de conception, y compris les conditions nécessaires et suffisantes pour leur existence.

De plus, ce chapitre étend sa couverture à l'intégration du contrôle tolérant aux défauts dans le cadre du contrôle basé sur l'observateur. La tolérance aux défauts garantit la robustesse et la résilience du système en détectant et en accommodant les défauts qui peuvent survenir pendant le fonctionnement. Le chapitre explore les techniques de détection, d'identification et d'accommodation des défauts, mettant en évidence leur intégration avec le contrôle basé sur l'observateur pour atteindre un contrôle tolérant aux défauts et une estimation simultanée de l'état.

Un observateur fonctionnel peut être utilisé comme contrôleur pour les systèmes LTI de descripteurs instables, ce qui peut être avantageux dans des applications où le coût computationnel a un grand impact sur les performances du système de contrôle. Par exemple, un observateur fonctionnel peut être utilisé dans de petites pompes à insuline, où les systèmes embarqués n'ont pas la pleine capacité d'un ordinateur.

D'autre part, l'approche généralisée de l'observateur fonctionnel permet de atténuer les effets des incertitudes paramétriques dans une certaine mesure.

L'une des principales avantages des observateurs fonctionnels est leur flexibilité sur l'estimation  $z(t)$ . Cette flexibilité ouvre des possibilités pour réaliser une stabilisation tolérante aux défauts simultanée, ainsi que le diagnostic et l'isolation de défauts. En exploitant cette caractéristique, les ingénieurs peuvent concevoir des systèmes de contrôle qui garantissent non seulement la stabilité et les performances du système en présence de défauts, mais qui fournissent également des informations précieuses sur la détection, l'identification et l'isolation des défauts.

La détection précoce des défauts facilitée par l'estimation des défauts peut conduire à une réduction de la durée d'arrêt du système et des coûts de maintenance. En détectant les défauts à leurs débuts, les activités de maintenance et de réparation peuvent être planifiées de manière proactive, minimisant les défauts inattendus du système et réduisant les temps d'arrêt coûteux. De plus, l'estimation des défauts peut aider à optimiser les efforts de maintenance en fournissant des informations sur la gravité et l'urgence des défauts.

Finalement, le chapitre 5, "Conclusions", présente les principales conclusions, contributions et implications de la recherche menée dans cette thèse sont résumées. Les objectifs principaux énoncés dans l'introduction sont revisités, et les limitations de la recherche sont discutées, tout en proposant des pistes pour des recherches futures dans le domaine.

## **Contributions**

Cette recherche a apporté des contributions significatives au domaine des observateurs fonctionnels et du contrôle pour les systèmes descripteurs. Les principales contributions sont les suivantes :

### 1. Développement d'Observateurs Fonctionnels Généralisés à Convergence en Temps Fini :

- Proposition et exploration du concept novateur d'observateurs fonctionnels généralisés à convergence en temps fini.
- Cette contribution étend le cadre traditionnel des observateurs fonctionnels en assurant la convergence dans un horizon temporel fini, offrant un outil précieux pour les applications soumises à des contraintes temporelles.

### 2. Observateurs Fonctionnels pour les Systèmes descripteurs Takagi-Sugeno :

- Développement d'observateurs fonctionnels spécifiquement adaptés aux systèmes descripteurs Takagi-Sugeno, largement utilisés pour modéliser des systèmes complexes avec une dynamique non linéaire.
- Les observateurs proposés offrent une approche systématique et efficace pour l'estimation d'état dans les systèmes descripteurs Takagi-Sugeno, tenant compte à la fois des modes continus et discrets.

### 3. Observateur Dynamique Généralisé avec Entrées Inconnues :

- Introduction du concept d'observateur dynamique généralisé avec entrées inconnues, permettant l'estimation à la fois des états du système et des entrées inconnues.
- Cette contribution améliore la capacité de l'observateur à gérer les entrées non mesurées, améliorant la précision de l'estimation d'état et la détection de défaillances dans les systèmes pratiques.

### 4. Contrôle basé sur l'Observateur pour les Systèmes Descripteurs :

- Exploration de l'intégration des observateurs fonctionnels dans le cadre du contrôle pour les systèmes descripteurs.
- En combinant l'estimation d'état et le contrôle dans un cadre unifié, notre recherche offre une solution complète pour obtenir un contrôle robuste et efficace en présence d'incertitudes et de perturbations.

Ces contributions font progresser l'état de l'art des observateurs fonctionnels et du contrôle pour les systèmes descripteurs, offrant des méthodologies et des techniques nouvelles pour résoudre des problèmes du monde réel difficiles. Les approches proposées étendent les fondements théoriques existants et fournissent des solutions pratiques avec des applications potentielles dans divers domaines.

### Produits

Les résultats de cette recherche ont donné lieu à quatre articles scientifiques acceptés et présentés lors de congrès nationaux et internationaux. Ces articles ont également été utilisés

pour publier un article dans la revue "Processes" sous le titre "Generalized Functional Observer for Descriptor Nonlinear Systems—A Takagi-Sugeno Approach".

### **Discussion**

La section de discussion examine en profondeur les résultats de la recherche, leur importance et leurs implications pour le domaine des observateurs fonctionnels et du contrôle pour les systèmes descripteurs. Elle souligne également les limites et les défis rencontrés lors du processus de recherche.

Les principales conclusions mettent en évidence la contribution significative de cette recherche au domaine des observateurs fonctionnels et du contrôle pour les systèmes descripteurs. Le développement d'observateurs fonctionnels généralisés à convergence en temps fini garantit la convergence dans un horizon temporel fini, abordant des applications soumises à des contraintes temporelles. Les observateurs fonctionnels adaptés aux systèmes descripteurs Takagi-Sugeno ont des implications directes pour les industries traitant de systèmes complexes présentant une dynamique non linéaire. L'intégration des observateurs fonctionnels dans le cadre du contrôle pour les systèmes descripteurs offre des avantages pratiques pour améliorer la performance du système, la fiabilité et la tolérance aux défaillances.

### **Implications Pratiques**

Les implications de cette recherche sont significatives pour des applications du monde réel dans divers domaines industriels et d'ingénierie. Les méthodologies et techniques développées offrent des solutions pratiques pour améliorer la performance des systèmes, la fiabilité et la tolérance aux défaillances dans des scénarios pratiques.

### **Limitations et travaux futurs**

Il est important de reconnaître les limitations de cette recherche, notamment la simplification des modèles et des hypothèses. Des modèles et hypothèses simplifiés ont été utilisés pour faciliter le développement de stratégies de contrôle et de conceptions d'observateurs. Cependant, ces simplifications peuvent ne pas capturer entièrement la complexité des systèmes du monde réel.

Bien que cette recherche ait apporté des contributions significatives, plusieurs avenues prometteuses pour des recherches futures émergent. Des travaux futurs devraient se concentrer sur la gestion de la saturation des actionneurs et des capteurs, l'extension des méthodologies proposées aux systèmes non linéaires, et la prise en compte des effets du bruit dans les mesures de sortie. Ces efforts permettront de répondre aux défis complexes liés au contrôle des systèmes, à la tolérance aux défaillances et à la présence de bruit dans les mesures de sortie.

# Chapter 1

## Introduction

### Contents

---

<b>1.1</b>	<b>Background</b> . . . . .	<b>1</b>
<b>1.2</b>	<b>State of the art</b> . . . . .	<b>3</b>
1.2.1	Descriptor systems . . . . .	3
1.2.2	Observers . . . . .	8
<b>1.3</b>	<b>Objectives</b> . . . . .	<b>10</b>
1.3.1	General objective . . . . .	10
1.3.2	Specific objectives . . . . .	11
<b>1.4</b>	<b>Justification</b> . . . . .	<b>11</b>
<b>1.5</b>	<b>Originality</b> . . . . .	<b>11</b>
<b>1.6</b>	<b>Document layout</b> . . . . .	<b>12</b>

---

This chapter serves as a gateway to the exploration undertaken in this thesis, shedding light on the motivations driving the proposed research, examining recent advancements in the field, and elucidating the overarching objectives. Through a comprehensive review, we delve into the background that lays the foundation for the subsequent analyses. The state of the art is scrutinized, with a specific focus on descriptor systems and observers, offering a snapshot of the current landscape. Clear objectives and goals are outlined, providing a roadmap for the ensuing studies. Additionally, the justification for this research is articulated, emphasizing its relevance and potential impact. The chapter concludes by highlighting the original contributions of this work and providing an overview of the document's layout, preparing the reader for a detailed exploration of functional observers and control strategies for descriptor systems.

### 1.1 Background

State estimation has been an important field of study since the introduction of the concept by D. Luenberger in 1966 [Luenberger, 1966]. In numerous control systems, the full knowledge of

the state is needed for the controller design (see Figure 1.1), where  $y(t) \in \mathbb{R}^p$  is the measured output,  $u(t) \in \mathbb{R}^m$  is the controlled input,  $\hat{x}(t) \in \mathbb{R}^n$  is the estimated state, and  $u_d(t) \in \mathbb{R}^m$  is the reference input. In most cases the full state is not available for direct measurement thus requiring to estimate the non measurable state.

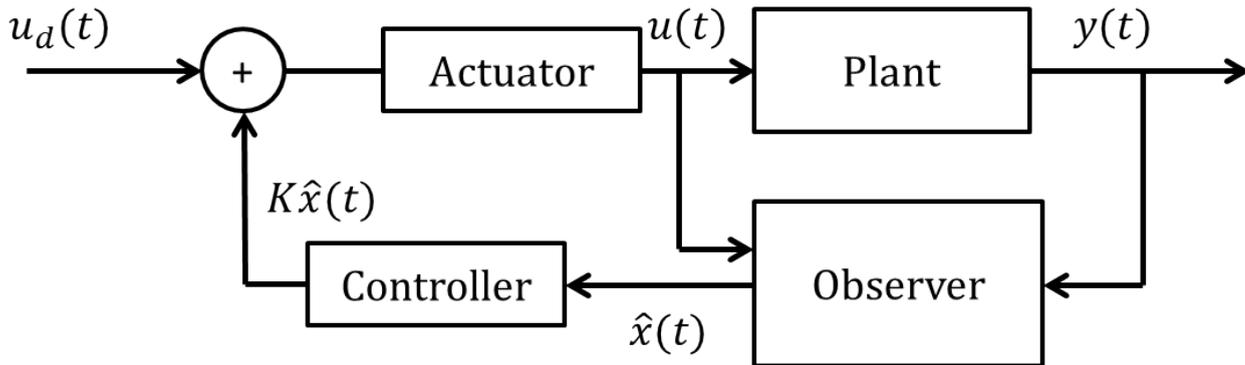


Figure 1.1: Classical observer-based control scheme with  $u(t) = K\hat{x}(t) + u_d(t)$

The main purpose of an observer is then to provide the estimation of the state for its use in the control system. This can be achieved in various forms, several observer structures have been proposed since its introduction. An observer can also be used as a powerful tool in control theory. The estimation of the non measurable state is just one of the many applications it can achieve. Depending on the structure of the observer and its design, it can be used to estimate faults, making it a powerful tool for fault diagnostic and fault-tolerant control theory. On the other hand, several structures of observers have been proposed through the years that can be adapted to numerous systems, such as LTI systems, non-linear systems, as well as descriptor and Takagi-Sugeno systems, making them a primary topic in control theory.

As stated above, numerous problems in the field of control theory and fault diagnosis can be solved by using the correct observer, however, this is a research area that can be explored further, new applications and observers can be developed, making them a source of studies for this thesis research.

Moreover, linear descriptor systems have been widely studied, they offer a more natural way to represent the behavior of dynamic systems. We can find dynamic systems in their descriptor form in a variety of fields such as mechanics, chemical processes, electrical and economics [Darouach, 2012]. Depending on the mathematical model of the plant, we can describe an LTI system prone to faults as a linear descriptor system, or we can include unknown inputs or measurement noise into the augmented descriptor system of the plant. Thus, descriptor linear systems offer a powerful tool to solve numerous control problems, using them to represent standard linear systems with unknown disturbances or faults.

## 1.2 State of the art

The state of the art in the field of functional observers and control for descriptor systems encompasses a diverse range of research and advancements. Over the years, significant progress has been made in developing robust observer designs, fault-tolerant control strategies, and integration techniques for handling uncertainties in real-world applications. Researchers have explored various approaches to enhance the performance, stability, and fault-tolerant behavior of descriptor systems, taking into account complex dynamics, nonlinearities, and practical implementation considerations. By reviewing the existing literature and examining the latest developments, this section provides an overview of the current state of the art in functional observers and control for descriptor systems, laying the foundation for the subsequent exploration of novel methodologies and advancements in this field.

### 1.2.1 Descriptor systems

Descriptor systems can represent a class of systems which can be described by dynamic equations as well as algebraic equations as seen in [Luenberger, 1978] where the author shows some concepts of descriptor linear discrete-time systems of the form

$$\begin{aligned}Ex(k + 1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k)\end{aligned}\tag{1.1}$$

such as the solvability, conditionability and the canonical form for solvable systems. The fact that we can represent systems that include differential and algebraic equations is a powerful tool in control systems. This advantage is the main motivation to design in a descriptor framework several modern observation and control algorithms which can be applied to a wide variety of physical processes such as hydraulic systems like the one shown in [Araujo et al., 2012] where an observer for the system consisting in three interconnected water tanks (see Figure 1.2) is designed.

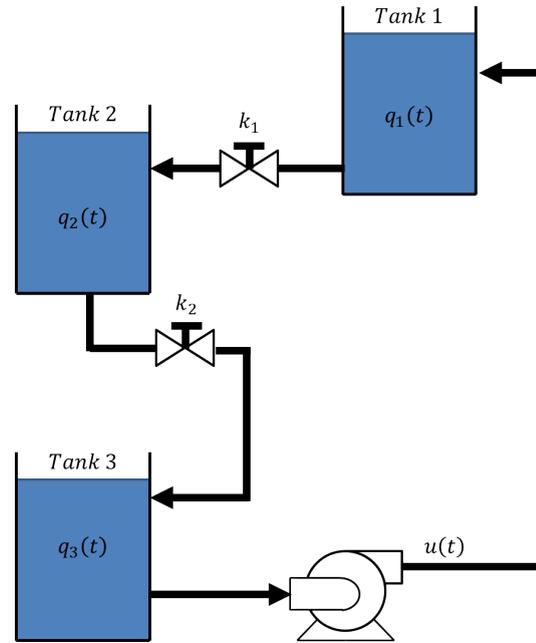


Figure 1.2: Interconnected water tanks used in [Araujo et al., 2012]

The system is modeled in a discrete-time descriptor linear system of the form (1.1). Biomechanical systems have also been studied using a descriptor approach. For example the work of [Guelton et al., 2008], where the human body is modeled as a double pendulum as shown in Figure 1.3, its dynamical behavior is described by a descriptor nonlinear system of the form

$$E(x(t))\dot{x} = A(x(t))x(t) + Bu(t) \quad (1.2)$$

where  $x(t)^T = [\theta_1 \ \theta_2 \ \dot{\theta}_1 \ \dot{\theta}_2]$ .

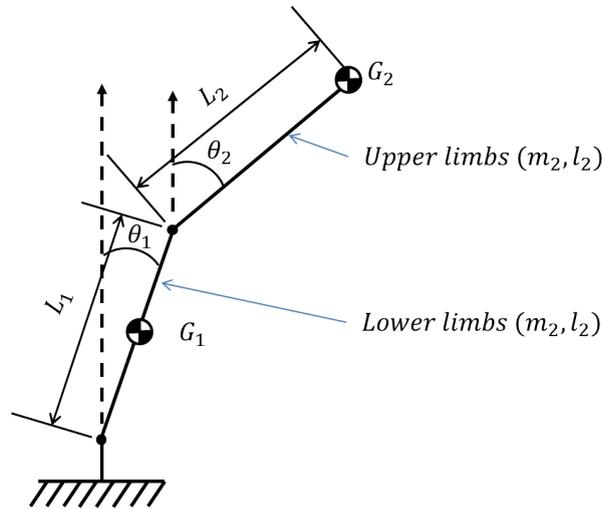


Figure 1.3: Human body stance represented as a double pendulum

In [Lewis, 1986], a survey on descriptor systems is made. Some historical background is given as well as the theories on frequency and time domain for the control of descriptor systems. Some fundamental concepts of descriptor systems, such as regularizability, controllability and observability have been studied in [Cobb, 1984, Zhang, 2006].

In [Kučera and Zagalak, 1988], the authors study the stability and stabilization of descriptor systems. They propose to stabilize a system of the form

$$E\dot{x}(t) = Fx(t) + Gu(t) \quad (1.3)$$

using state feedback with a control law of the form  $u(t) = -Kx(t)$  and determining the limits of the state feedback in altering the dynamical behavior of (1.3). Some restrictions of their work are that the matrix pencil  $sE - A$  is assumed to be regular, strictly proper, and regular. The results of the studies are shown using the Kronecker form.

$$\begin{aligned} \dot{x}_1(t) &= F_1x_1(t) + G_1u(t) \\ E_2\dot{x}_2(t) &= x_2(t) + G_2u(t) \end{aligned} \quad (1.4)$$

The state feedback control has been studied by [Dodig and Stojić, 2009], where a class of singular systems of the form

$$E\dot{x}(t) = Ax(t) + Bu(t),$$

where  $x(t)$  is the state and  $u(t)$  is the input is considered. The control signal is given by

$$u(t) = Kx(t) + v(t),$$

where  $v(t)$  is an external input. In this paper, the stabilization of the system is achieved by using matrix pencil theory with the canonical form of the singular system.

Faulty systems can be represented in a descriptor form as seen in [Lan and Patton, 2015], where they represent a descriptor system of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + Ff(t) + Dd(t) \\ y(t) &= Cx(t) \end{aligned} \quad (1.5)$$

where  $f(t) \in \mathbb{R}^r$  and  $d(t) \in \mathbb{R}^l$  are the fault and the unknown disturbance, respectively. Some restrictions to their work is that the fault is considered constant, i.e.  $\dot{f}(t) = 0$ . The system (1.5) can be represented as

$$\begin{aligned} \bar{E}\dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{B}u(t) + \bar{D}d(t) \\ y(t) &= \bar{C}\bar{x}(t) \end{aligned} \quad (1.6)$$

where  $\bar{x}^T(t) = [x(t) \ f(t)]$ ,  $\bar{E} = \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}$ ,  $\bar{A} = \begin{bmatrix} A & F \\ 0 & 0 \end{bmatrix}$ ,  $\bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$ ,  $\bar{D} = \begin{bmatrix} D \\ 0 \end{bmatrix}$  and  $\bar{C} = [C \ 0]$ . The system (1.6) is then used to design a functional observer that estimates a control law capable to stabilize the system (1.5).

In [Gu and Yang, 2017], a class of nonlinear systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + B_d d(t) + G\psi(t, x(t), u(t)) \\ y(t) &= Cx(t) + D_f f(t) \end{aligned} \quad (1.7)$$

is represented in a descriptor form as

$$\begin{aligned} \bar{E}\dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{B}u(t) + \bar{B}_d d(t) + \bar{G}\psi(t, \bar{x}(t), u(t)) + \bar{D}_h h(t) \\ y(t) &= \bar{C}\bar{x}(t) \end{aligned} \quad (1.8)$$

where  $\bar{x}^T(t) = [x(t) \ h(t)]$ ,  $h(t) = D_f f(t)$ ,  $\bar{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ ,  $\bar{A} = \begin{bmatrix} A & 0 \\ 0 & -I \end{bmatrix}$ ,  $\bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$ ,  $\bar{B}_d = \begin{bmatrix} B_d \\ 0 \end{bmatrix}$ ,  $\bar{D}_h = \begin{bmatrix} 0 \\ I \end{bmatrix}$ ,  $\bar{G} = \begin{bmatrix} G \\ 0 \end{bmatrix}$  and  $\bar{C} = [C \ I]$ . The equivalent system (1.8) is then used to design an observer that estimates the state of the system and the sensor faults.

The state of the art in descriptor LTI systems encompasses several key areas of research and development. As we conclude our exploration of the existing body of knowledge, it becomes evident that these areas form a dynamic landscape with ongoing advancements and challenges. This comprehensive overview serves as the foundation upon which our research builds, acknowledging the strides made by predecessors while identifying gaps and opportunities for further investigation. Based on this investigation, the state of the art in descriptor LTI systems encompasses several key areas of research and development:

- Modeling and Representation:
  - The state-space representation of descriptor systems, also known as the descriptor form or generalized state-space form, has been widely studied. This representation explicitly incorporates the differential and algebraic equations governing the system's behavior.

- Various methods have been proposed to transform descriptor systems into regular (non-descriptor) state-space form or to simplify and normalize their equations.
- System identification techniques for descriptor LTI systems have been explored, allowing the estimation of system parameters from input-output data.
- Stability and Control Analysis:
  - Stability analysis for descriptor systems is a fundamental aspect of control design. Researchers have developed techniques to analyze the stability properties of descriptor systems, including the investigation of stability conditions and the development of stability criteria.
  - Robust control design methodologies have been developed to handle uncertainties in descriptor systems, such as parametric uncertainties, disturbances, and noise.
  - Lyapunov-based methods, such as Lyapunov equations and Lyapunov stability theory, have been extended to address stability analysis and controller synthesis for descriptor LTI systems.
- Observer Design and State Estimation:
  - Observer design for descriptor LTI systems has been a subject of extensive research. Various observer-based techniques have been proposed, including full-order and reduced-order observers, H-infinity observers, and Kalman-like filters.
  - Robust and fault-tolerant observer design methodologies have been developed to handle uncertainties, disturbances, and faults in descriptor systems.
  - Functional observers, a specific class of observers, have gained attention for their ability to estimate both the states and unknown inputs of descriptor systems.
- Control Design and Optimization:
  - Control design for descriptor LTI systems has been investigated, including the design of state feedback controllers, output feedback controllers, and optimal control strategies.
  - Model Predictive Control (MPC) techniques have been adapted and extended to address the control of descriptor systems.
  - Optimization-based control strategies, such as linear programming and quadratic programming, have been employed to handle constraints and achieve desired system performance.
- Applications and Practical Implementations:
  - Descriptor LTI systems find application in diverse fields, including robotics, power systems, chemical processes, and aerospace engineering.

- Real-world case studies and practical implementations of control strategies for descriptor systems have been reported, demonstrating their effectiveness and applicability in industrial settings.

The state of the art in descriptor LTI systems is continuously evolving, with ongoing research focusing on advanced control methodologies, fault diagnosis and accommodation, optimization techniques, and the integration of descriptor models with other control paradigms. This active research area offers exciting opportunities for advancements in control theory and practical applications.

## 1.2.2 Observers

Observers have been used since the introduction of the concept by D. Luenberger [Luenberger, 1966, Luenberger, 1971]. They are useful as virtual sensors where part of the state is not available for direct measurement by physical sensors. State estimators for linear time-invariant systems have been widely studied [Ali et al., 2015]. On the other hand, the full state is not always needed to be estimated since we only need the state that is not available for direct measurement [Trinh and Fernando, 2011]. This can be achieved with a functional observer which estimates a linear combination of the state [Fernando et al., 2010, Fernando et al., 2011], this is useful in control theory since we can estimate a control signal capable of stabilize an unstable system.

It is known that conventional proportional observers are sensitive to parametric uncertainties and disturbances, this leads to an increase in the estimation error in steady state. Proportional-Integral observers have been proposed to solve this problem, as shown by [Shafai and Carroll, 1985] who designed an adaptive Luenberger-based observer to estimate the state of a dynamical system with unknown disturbances of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Dd(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1.9}$$

In order to estimate the state, and the unknown disturbance, an adaptive observer of the form

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + D\hat{d}(t) + K_p(y(t) - C\hat{x}(t)) \\ \dot{\hat{d}}(t) &= K_I(y(t) - C\hat{x}(t)) \end{aligned} \tag{1.10}$$

is proposed.

The observer (1.10) is then used with a quadratic regulator to stabilize a linear dynamical system in [Shafai et al., 2002].

There have been numerous authors that propose methods to design functional observers for Linear Time-Invariant (LTI) systems as in [Zhang, 1987]. Here, the authors propose the design of a functional observer using the frequency domain analysis. In [Yao et al., 1997], a functional observer design for time delayed systems is proposed using the frequency domain analysis. The stabilization of standard linear systems has been studied in [Aldeen and Trinh, 1999] where the

authors propose a functional observer to stabilize an unstable standard linear system of the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}\tag{1.11}$$

They propose a control law  $u(t) = Kx(t)$  and decompose the feedback gain matrix as  $K = NT + WC$ . The estimated control law becomes then  $\hat{u}(t) = Pz(t) + Qy(t)$ , where  $z(t) = Tx(t)$  is an auxiliary state of the observer. By proposing  $\dot{z}(t) = Nz(t) + TBu(t) + Fy(t)$ , the problem of the observer design is reduced to determine the gain matrices  $P$ ,  $Q$ ,  $N$ ,  $T$  and  $F$ .

State estimation for descriptor systems have been studied by [Darouach and Boutayeb, 1995, Darouach et al., 1996, Darouach, 2000, Darouach, 2001, Müller and Hou, 1991] where the authors propose different structures of observers for descriptor linear systems of the form

$$\begin{aligned}E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}\tag{1.12}$$

More recently, [Darouach, 2012] shows a more generalized structure of functional observer for descriptor systems. A functional observer of the form

$$\begin{aligned}\dot{\zeta}(t) &= N\zeta(t) + F \begin{bmatrix} E^\perp Bu(t) \\ y(t) \end{bmatrix} + Hu(t) \\ \hat{z}(t) &= P\zeta(t) + Q \begin{bmatrix} E^\perp Bu(t) \\ y(t) \end{bmatrix}\end{aligned}\tag{1.13}$$

In [Osorio-Gordillo et al., 2016], the authors propose a design method for Generalized Dynamic Observers (GDO) for descriptor LTI systems of the form (1.12). The observer structure is constructed as it follows

$$\begin{aligned}\dot{\zeta}(t) &= N\zeta(t) + Jv(t) + F \begin{bmatrix} E^\perp Bu(t) \\ y(t) \end{bmatrix} + Hu(t) \\ \dot{v}(t) &= S\zeta(t) + Gv(t) + M \begin{bmatrix} E^\perp Bu(t) \\ y(t) \end{bmatrix} \\ \hat{z}(t) &= P\zeta(t) + Q \begin{bmatrix} E^\perp Bu(t) \\ y(t) \end{bmatrix}.\end{aligned}\tag{1.14}$$

In [Osorio-Gordillo et al., 2015], the observer is used to estimate the state of a descriptor system with unknown disturbances. The observer is designed under the  $H_\infty$  optimal constraints. Finite-time convergence, is not restricted to sampling and discrete time techniques, thus, finite-time convergence is an appealing topic of study [Engel and Kreisselmeier, 2002]. This is specially important in fault estimation because a large number of practical control system require short response constraints. Designing an observer capable to reach correct convergence within a finite-time constraint can be a powerful tool for fault diagnosis.

Observers play a crucial role in control systems by estimating the unmeasured states of a system based on available measurements. Over the years, various observer designs have been proposed, including well-known techniques such as Luenberger observers, Kalman filters, and extended Kalman filters. Each of these designs has its own set of advantages and limitations, and researchers have focused on developing observer methodologies that can handle nonlinearities, uncertainties, and disturbances encountered in real-world systems. Advanced observer techniques, such as sliding mode observers, high-gain observers, and adaptive observers, have been extensively investigated to improve estimation accuracy and robustness. Moreover, observer-based control strategies, including state feedback control and output feedback control, have been developed to achieve desired system performance based on the estimated states.

Functional observers have emerged as a specific class of observers that estimate not only the states but also the unknown inputs or disturbances affecting a system. They have garnered significant attention for their capability to estimate unmeasured variables or disturbances that directly impact the system dynamics. Researchers have focused on the design and analysis of functional observers, delving into their theoretical foundations and exploring their application to a wide range of control problems. Various types of functional observers have been proposed, such as proportional-integral (PI) observers, sliding mode observers, and high-gain observers. Furthermore, the integration of functional observers with control strategies, including fault-tolerant control and adaptive control, has been extensively investigated to enhance system performance and robustness in the face of uncertainties. The development of robust and fault-tolerant functional observers has also been a subject of interest, aiming to handle uncertainties, disturbances, and faults in real-world systems. Practical implementations and case studies have demonstrated the effectiveness of functional observers in diverse applications, ranging from power systems to robotics and automotive control.

In summary, the state of the art in observers and functional observers is a dynamic field of research that continues to evolve. Ongoing studies are focused on advanced observer designs, robustness analysis, fault diagnosis and accommodation, nonlinear and adaptive observer techniques, and the integration of functional observers with control strategies. These research efforts open up new possibilities for advancements in state estimation, system monitoring, and control across various applications, creating opportunities to improve the performance and robustness of real-world systems.

## 1.3 Objectives

### 1.3.1 General objective

To obtain the estimation of a linear combination of the system states by designing a functional observer. The linear function can be considered as the states of the system or as a state feedback controller.

### 1.3.2 Specific objectives

- To establish the theoretical foundations and mathematical principles underlying functional observers for accurate state estimation.
- To develop methodologies for designing functional observers tailored to descriptor systems.
- Investigate the necessary conditions for the existence of functional observers and propose design guidelines.
- To explore fault detection and fault diagnosis techniques suitable for descriptor systems.
- To develop strategies for fault accommodation and fault-tolerant control based on the information provided by the functional observers.
- To conduct extensive simulations to assess the performance of the developed methodologies in various scenarios.

## 1.4 Justification

In numerous control systems, the complete state of the system is generally not needed. State feedback, for example, require a linear combination of the state. Functional observers are useful since they can be used to estimate a linear combination of the state of a system. This advantage is used to design reduced order observers for the state and fault estimation in control systems. Functional observers can be used to design reduced order observer-based control systems and fault-tolerant control systems.

This research project stems from the growing demand for advanced control techniques that can address the complex dynamics and uncertainties encountered in practical systems. The existing literature reveals the need for robust state estimation methods and fault-tolerant control strategies that can enhance system performance, reliability, and adaptability. By undertaking this research project, we aim to bridge the existing gaps and contribute to the field by developing innovative methodologies that can accurately estimate system states, accommodate faults, and enable robust and efficient control in descriptor systems. The outcomes of this research project have the potential to make significant contributions to various industrial domains and enhance the understanding and practical applicability of functional observers and control techniques for descriptor systems.

## 1.5 Originality

Contrary to previous works, this thesis takes advantage of the generalized form of state observers to propose a form and design of a functional observer for descriptor systems. The results are then extended to applications such that finite-time estimation and systems prone to parametric uncertainties.

While significant progress has been made in control theory, this research project introduces an innovative methodology and approach that address specific challenges and fill critical gaps in the current literature. By combining advanced techniques and leveraging recent developments, this project offers a fresh perspective and novel solutions for accurate state estimation, fault-tolerant control, and system adaptability in descriptor systems. The originality of this research lies in the development of new algorithms, theoretical frameworks, and integration strategies that go beyond the existing approaches, providing novel insights and pushing the boundaries of knowledge in the field. By presenting original contributions, this research project aims to advance the scientific community's understanding of functional observers and control for descriptor systems, offering unique and valuable perspectives that have the potential to significantly enhance the performance and robustness of these systems.

## 1.6 Document layout

This document is divided in four parts:

- **Chapter I.** This chapter lays the groundwork by elucidating the objectives and goals of the thesis proposal. It delves into the justification that propelled the exploration of the chosen topic, providing a comprehensive overview of the motivations behind the research.
- **Chapter II.** Titled “Theoretical Framework,” this chapter serves as a comprehensive exploration of fundamental concepts and the current state of the art related to the topics under study. It establishes a theoretical foundation essential for the subsequent analyses and discussions.
- **Chapter III.** In this chapter, titled “Functional Observers for LTI Systems,” the focus is on the design of Functional Observers tailored for LTI descriptor systems. The chapter extends its scope to encompass systems with unknown inputs, and culminates in an examination of state and fault estimation for Takagi-Sugeno systems.
- **Chapter IV.** Titled “Application of Functional Observers,” this chapter explores the potential practical implementations of functional observers. It covers observer-based control for descriptor LTI systems, its utilization in fault detection and estimation, and introduces the concept of observer-based fault-tolerant control.
- **Chapter V.** The final chapter, “Conclusion,” consolidates the findings and insights gleaned throughout the research. It encapsulates general conclusions drawn from the study and outlines potential avenues for future research, providing a comprehensive closure to the thesis.

# Chapter 2

## Theoretical Framework

### Contents

---

<b>2.1</b>	<b>Descriptor systems</b>	<b>14</b>
2.1.1	Examples	14
2.1.2	Descriptor systems properties	18
<b>2.2</b>	<b>LMI regions</b>	<b>24</b>
2.2.1	$\mathcal{D}$ -stability	24
<b>2.3</b>	<b>Takagi-Sugeno systems</b>	<b>25</b>
<b>2.4</b>	<b>Functional observers</b>	<b>26</b>
<b>2.5</b>	<b>Conclusion</b>	<b>27</b>

---

This chapter serves as the foundation for the research conducted in this thesis on functional observers and control for descriptor systems. It presents a comprehensive overview of the theoretical underpinnings and key concepts essential for understanding the subsequent chapters. By delving into the fundamental principles and mathematical tools used in the analysis and design of functional observers and control strategies, this chapter establishes a solid theoretical framework. Through this comprehensive theoretical framework, this chapter sets the stage for the subsequent chapters to delve into more specific aspects of functional observers and control for descriptor systems, providing the necessary theoretical groundwork to advance the understanding and application of these techniques in practical scenarios.

It begins with a detailed examination of descriptor systems, unraveling their intricacies and delving into examples that illuminate their relevance in practical applications. Properties intrinsic to descriptor systems are then scrutinized, shedding light on key characteristics that form the basis for subsequent discussions. Moving forward, the exploration extends to Linear Matrix Inequality (LMI) regions, with a particular focus on  $\mathcal{D}$ -stability, providing a mathematical lens through which system stability can be assessed.

The chapter further advances into Takagi-Sugeno systems, exploring their unique characteristics and applicability in handling complex, nonlinear systems. Finally, the theoretical framework encompasses functional observers, elucidating their role in system observation and

control. By establishing a firm theoretical groundwork, this chapter propels the subsequent sections to delve into more nuanced aspects of functional observers and control for descriptor systems. This groundwork is essential for advancing the comprehension and practical application of these techniques in real-world scenarios, providing a comprehensive toolkit for researchers and practitioners alike.

## 2.1 Descriptor systems

Descriptor systems are a class of systems that include the dynamic and static behavior of a plant. Mathematically, they are represented by a set of dynamic and algebraic equations. Descriptor systems are a powerful modeling tool since they can describe behaviors that standard dynamical systems cannot.

Nonlinear descriptor systems can be represented by the following set of equations:

$$\begin{aligned} E\dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= g(x(t)) \end{aligned} \tag{2.1}$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $u(t) \in \mathbb{R}^m$  is the input and  $y(t) \in \mathbb{R}^p$  is the measured output.  $f(\cdot)$  and  $g(\cdot)$  are nonlinear continuous and infinitely differentiable functions.  $E$  is a singular constant matrix with appropriate dimensions. If system (2.1) is linearized, the result in continuous case is as it follows.

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{2.2}$$

Linear descriptor systems have features that does not exist or are different than standard LTI systems. Impulsive behavior and impulsive modes are important features of descriptor systems and need to be solved in order to design a control system. Concepts as regularity, reachability, controllability and observability will be presented in this section the concepts are extracted form [Duan, 2010] and [Dai, 1989].

### 2.1.1 Examples

Descriptor modeling can be used to describe the interaction between different systems, for example, in 2.1, we can see the electric circuit of an electrical motor and its interaction in a mechanical rotary system.

## Electrical motor

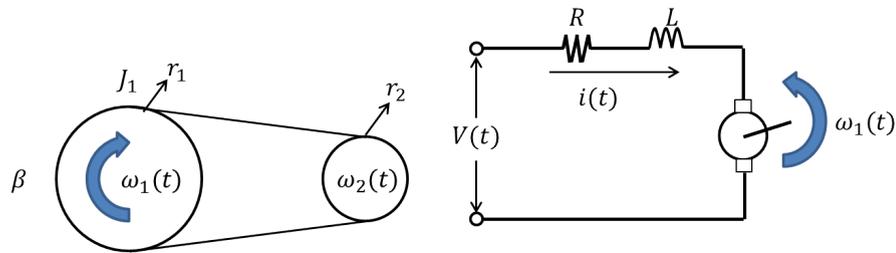


Figure 2.1: Electrical - Mechanical system

From Figure 2.1 we can derive three equations: a dynamic equation from the mechanical system, another dynamical equation from the motor circuit, and an algebraic equation that couples both systems.

$$\begin{aligned}
 J\dot{\omega}_1(t) &= -\beta\omega_1(t) + K_T i(t) \\
 L\dot{i}(t) &= -K_b\omega_1(t) - Ri(t) + V(t) \\
 r_1\omega_1(t) - r_2\omega_2(t) &= 0
 \end{aligned} \tag{2.3}$$

$$\begin{bmatrix} J & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\omega}_1(t) \\ \dot{i}(t) \\ \dot{\omega}_2(t) \end{bmatrix} = \begin{bmatrix} -\beta & K_T & 0 \\ -K_b & R & 0 \\ r_1 & 0 & -r_2 \end{bmatrix} \begin{bmatrix} \omega_1(t) \\ i(t) \\ \omega_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} V(t) \tag{2.4}$$

## Linearized single-link robot arm

In this case, we present a single-link robot arm which is connected to an electrical motor (as it can be seen in Figure 2.2).

The mathematical model that represents the system is:

$$\begin{aligned}
 J\ddot{\theta}(t) &= -mgl_c \sin(\theta(t)) - \beta\dot{\theta}(t) + lF_l + d(t) \\
 F_l &= C_s\omega(t)^2
 \end{aligned} \tag{2.5}$$

where:

- $\theta(t)$  is the position of the arm in *rad*.
- $F_l(t)$  is the lift force in *N*.
- $d(t)$  is an unknown input.
- $J$  is the moment of inertia of the pendulum in  $Kg/m^2$ .
- $m$  is the mass of the arm in  $Kg$ .

- $l_c$  is the distance from the pivot point to the center of gravity in  $m$ .
- $\beta$  is the viscous coefficient of friction of the pivot point in  $Nms/rad$ .
- $l$  is the length of the arm in  $m$ .
- $C_s$  is a lift constant in  $Kg * m$ .
- $g$  is the acceleration of gravity in  $m/s^2$ -

### Takagi-Sugeno model

The mathematical model of single link robot arm (see Figure 2.2).

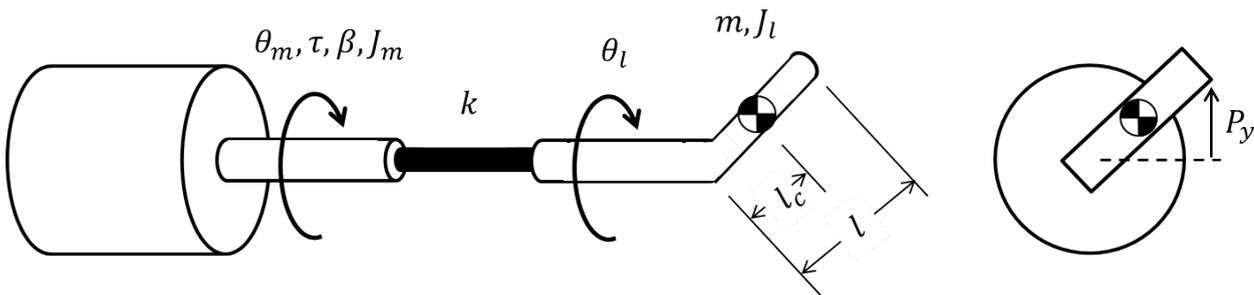


Figure 2.2: Single link robot arm

The model was selected from [Chakrabarty et al., 2017] with the following nonlinear model:

$$\begin{aligned} J_m \ddot{\theta}_m(t) &= -\beta \dot{\theta}_m(t) - k(\theta_m(t) - \theta_l(t)) + K_\tau u(t) \\ J_l \ddot{\theta}_l(t) &= k(\theta_m(t) - \theta_l(t)) + l_c m g \sin \theta_l(t) \end{aligned} \quad (2.6)$$

where:

- $\theta_m$  is the angular position of the motor shaft.
- $\theta_l$  is the angular position of the link.
- $J_m$  and  $J_l$  are the moment of inertia of the motor shaft and the link, respectively.
- $\beta$  is the viscous coefficient of friction of the motor.
- $k$  is the spring rigidity coefficient.
- $m$  is the mass of the link.
- $l_c$  is the distance of the center of mass.
- $l$  is the length of the link.

- $K_\tau$  is the constant of the motor.

Taking as the states  $x_1(t) = \theta_m(t)$ ,  $x_2(t) = \dot{\theta}_m(t)$ ,  $x_3(t) = \theta_l(t)$ ,  $x_4(t) = \dot{\theta}_l(t)$  and  $x_5(t) = d_y(t)$ , we can represent (2.6) as it follows:

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -\frac{\beta}{J_m}x_2(t) - \frac{k}{J_m}x_1(t) + \frac{k}{J_m}x_3(t) + \frac{K_\tau}{J_m}u(t) \\
\dot{x}_3(t) &= x_4(t) \\
\dot{x}_4(t) &= \frac{k}{J_l}x_1(t) - \frac{k}{J_l}x_3(t) + \frac{l_c mg}{J_l} \sin x_3(t) \\
0 &= l \sin x_3(t) - x_5(t)
\end{aligned} \tag{2.7}$$

its representation as a nonlinear descriptor system is

$$E\dot{x} = f(x, t) + Bu(t) \tag{2.8}$$

$$\text{where } E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{K_\tau}{J_m} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ and } f(x, t) = \begin{bmatrix} x_2(t) \\ -\frac{\beta}{J_m}x_2(t) - \frac{k}{J_m}x_1(t) + \frac{k}{J_m}x_3(t) \\ x_4(t) \\ \frac{k}{J_l}x_1(t) - \frac{k}{J_l}x_3(t) + \frac{l_c mg}{J_l} \sin x_3(t) \\ l \sin x_3(t) - x_5(t) \end{bmatrix}.$$

We chose as the scheduling variable  $\rho(t) = \frac{\sin x_3(t)}{x_3(t)}$ , so the system (2.7) can be represented:

$$\begin{aligned}
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{k}{J_m} & -\frac{\beta}{J_m} & \frac{k}{J_m} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{k}{J_l} & 0 & \frac{1}{J_m}(k - l_c mg\rho(t)) & 0 & 0 \\ 0 & 0 & l\rho(t) & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{K_\tau}{J_m} \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} x(t)
\end{aligned} \tag{2.9}$$

The weighting functions for the T-S systems are:

$$w_1(t) = \frac{\bar{\rho} - \rho(t)}{\bar{\rho} - \underline{\rho}} \tag{2.10}$$

$$w_2(t) = 1 - w_1(t) \tag{2.11}$$

The Takagi-Sugeno system can be then represented as it follows:

$$\begin{aligned}
E\dot{x}(t) &= \sum_{i=1}^2 w_i(t) (A_i x(t) + Bu(t)) \\
y(t) &= Cx(t)
\end{aligned} \tag{2.12}$$

with

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -30 & -5 & 30 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 3.75 & 0 & -8.33 & 0 & 0 \\ 0 & 0 & 0.18 & 0 & -1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -30 & -5 & 30 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 3.75 & 0 & -2.77 & 0 & 0 \\ 0 & 0 & -0.04 & 0 & -1 \end{bmatrix}$$

## 2.1.2 Descriptor systems properties

In this section, the fundamental properties of the descriptor systems are discussed.

### Regularity

For a standard LTI system there exists always an unique solution, this is however not always the case for descriptor systems. A solution for a descriptor system can be non existent or can have multiple solutions. The conditions leading to the existence and uniqueness of a solution result in the concept of regularity. Regularity is a very important property of a descriptor linear system since it guarantees the existence and uniqueness of a solution.

**Definition 2.1.1** *A system of the form (2.2) is said to be regular if there exists a scalar  $\gamma \in \mathbb{C}$  such that the following condition holds.*

$$\det(\gamma E - A) \neq 0 \quad (2.13)$$

*Equation (2.13) is equivalent to say that the polynomial equation  $\det(sE - A)$  is not equal to zero. In this case we say that the pair  $(E, A)$  is regular.*

Regularity also introduces the concept of regularizability which is an important topic in control theory since its goal is to present the conditions for the existence of a controller that guarantees that the closed loop system is regular.

### Stability

Stability is a key factor in any control system. It is well known that a practical control system should be stable, otherwise, it can lead to the incorrect behavior of the system or even its destruction.

We consider now a class of systems of the form

$$E\dot{x}(t) = Ax(t), \quad x(0) = x_0 \quad (2.14)$$

We assume that (2.14) is regular since it has an unique solution and makes the definition easier. We know that the system is stable if  $\lim_{t \rightarrow \infty} x(t) = 0$ .

**Definition 2.1.2** *A descriptor linear system is said to be stable if and only if*

$$\sigma(sE - A) \subset \mathbb{C} = \{s | s \in \mathbb{C}, \operatorname{Re}(s) \leq 0\} \quad (2.15)$$

where  $\sigma(sE - A)$  denotes the poles of the pair  $(E, A)$ .

### Descriptor system transformation

Regular descriptor systems can be divided in two subsystems, a slow system and a fast system. The slow system is a set of ordinary differential equations with an unique solution and its response is determined by the initial conditions and the input signal  $u(t)$ , the fast system can have impulsive behaviors. Both systems, the slow and the fast can be obtained using a singular value decomposition which tells us that there exist two matrices  $P$  and  $Q$  such that.

$$\begin{aligned} \tilde{E} &= PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} \\ \tilde{A} &= PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \\ \tilde{B} &= PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \end{aligned} \quad (2.16)$$

where  $N \in \mathbb{R}^{n_2 \times n_2}$  is a nilpotent matrix with nilpotent index  $h$ . The system (2.2) is equivalent to the system

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t) \\ N \dot{x}_2 &= x_2(t) + B_2 u(t) \end{aligned} \quad (2.17)$$

### Impulsive-free behavior

Impulsive behaviors are usually undesired in practical applications. The impulsive term and its derivatives can saturate the state or even damage the system. Impulse-free descriptor systems are thus more practical for control applications. There exists various methods for eliminating the impulsive behavior, however, for these thesis studies, all descriptor systems presented are assumed to be impulse-free.

A regular descriptor system is said to be impulse-free if and only if one of the following conditions is true:

1. The nilpotent matrix  $N$  in its fast subsystem is a zero matrix.
2.  $\deg(\det(\lambda E - A)) = \operatorname{rank}(E)$
3. The system has  $\operatorname{rank}(E)$  finite poles.

## Admissibility

A descriptor system is said to be admissible if it is both: stable and impulse free. The admissibility of the descriptor linear system of the form

$$E\dot{x}(t) = Ax(t) \quad (2.18)$$

is also related to a type of generalized Lyapunov matrix equations, but in the following form:

$$E^T X A + A^T X E = -E^T Y E \quad (2.19)$$

where  $Y = Y^T < 0$ .

The admissibility of a descriptor system of the form (2.18) is given by the following equivalent statements:

- *If there exist  $X = X^T \geq 0$ , and  $Y = Y^T > 0$  such that LMI (2.19), then system (2.18) is admissible.*
- *If the pair  $(E, A)$  is admissible, then for every  $\gamma > 0$  there exist a matrix  $X > 0$  that satisfies (2.19)*

If we consider the system 2.19, then the the following statements are true:

- The pair  $(E, A)$  is said to be regular if and only if  $\det(\lambda E - A)$  is not identically 0.
- The pair  $(E, A)$  is said to be impulse-free if  $\deg(\det(\lambda E - A)) = \text{rank}(E)$
- The pair  $(E, A)$  is said to be stable if all the roots in  $(\lambda E - A) = 0$  have negative real parts.
- The pair  $(E, A)$  is said to be admissible if it is regular, impulse-free, and stable.

The following theorem gives the necessary conditions for system (2.19) to be admissible.

**Theorem 2.1.3** *The pair  $(E, A)$  is said to be admissible of and only if there exists a matrix  $P$ , such that*

$$\begin{aligned} EP_c^T &= P_c^T E \geq 0 \\ P_c^T A + A^T P_c &< 0 \end{aligned} \quad (2.20)$$

or

$$\begin{aligned} EP_c &= P_c^T E^T \geq 0 \\ AP_c + P_c^T A^T &< 0 \end{aligned} \quad (2.21)$$

### Controllability and observability

In general, descriptor systems are fixed into algebraic constraints, the state of a dynamical system is then bounded. A vector  $w(t) \in \mathbb{R}^n$  is said to be reachable if there exists an initial condition  $x(0) = x_0$  and an admissible input  $u(t) \in \mathbb{R}^m$  such that the state satisfies  $x(t_1) = w(t)$  in a finite time  $t_1 > 0$ . The set that contains all the reachable vectors is called reachable set and it is defined as

$$\mathcal{R}_t[x_{10}] := \{w \mid \exists u(t) \in \mathbb{R}^m : x(t, u, x_{10}) = w\} \quad (2.22)$$

the reachable set  $\mathcal{R}_t[x_{10}]$  is the set of reachable vectors at a time  $t$  from an initial state  $x_1(0) = x_{10}$ .

Reachability is an important concept since the controllability and observability of a descriptor system are associated with its reachability. As mentioned earlier, controllability and observability are important features of a system. In standard systems, for example, the poles of a feedback system can be arbitrarily assigned if and only if the system is controllable, similarly, the poles of a full order observer for a standard system can be arbitrarily assigned if and only if the system is observable.

On the other hand, there exist multiple definitions of controllability and observability for descriptor linear systems. These definitions are based on the controllability and observability of the slow and the fast system in a descriptor system of the form (2.17).

**C-Controllability** A descriptor system of the form (2.17) is said to be *completely controllable*, or C-Controllable, if there exists a finite time  $t_1 > 0$ , an initial value  $x_0 \in \mathbb{R}^n$  and an objective vector  $w \in \mathbb{R}^n$  for which there is a control input  $u(t)$  that satisfies  $x(t_1, u, x_0) = w$ . This means that for any initial condition  $x_0$  we can find a control input  $u(t)$  such that the state of the system is driven to any desired position in  $\mathbb{R}^n$  in a finite time. This leads to the following definition.

**Definition 2.1.4** *A regular descriptor system of the form (2.17) is C-Controllable if and only if*

$$\mathcal{R}_t[0] = \mathbb{R}^n, \forall t > 0 \quad (2.23)$$

For a regular descriptor system, it is necessary that both, the slow and the fast system are C-Controllable.

**R-Controllability** A regular descriptor system is said to be *controllable in a reachable set*, or R-Controllable, if for any initial condition  $x_0$  it is possible to reach any state within the set  $\mathcal{R}$  in a finite time  $t_1 > 0$  with a control input  $u(t)$ .

Generally the descriptor systems are limited by algebraic constraints, this means that, in general, descriptor systems cannot be C-Controllable. The following definition applies

**Definition 2.1.5** *A regular descriptor system of the form (2.17) is said to be R-Controllable if*

$$\mathcal{R}_t[0] = \mathcal{R}, \forall t > 0 \quad (2.24)$$

where  $\mathcal{R}_t$  represents the reachable set for an initial condition 0.

**I-Controllability** The concepts of C-Controllability and R-Controllability are only concerned in the steady state behavior of a descriptor system, yet, it is known that impulsive behaviors may exist at  $t = 0$  in descriptor systems. Impulsive behavior can lead to malfunction of the control system, so it is necessary to eliminate the unwanted impulsive behaviors. This introduces the concept of I-Controllability, this is the ability to cancel all impulsive behaviors using an admissible control input  $u(t)$ . This can be reached by making the response of system finite the moment of the impulse occurs. We propose now the following definition.

**Definition 2.1.6** *A regular descriptor system of the form (2.17) is I-Controllable if and only if for any vector  $x_{20} \in \mathbb{R}^n$ , there exists an admissible control input  $u(t)$  such that*

$$Nx_{20} + [NB_2 + N^2B_2 + \dots + N^{h-1}B_2] \begin{bmatrix} u(0) \\ \dot{u}(0) \\ \vdots \\ u^{(h-1)}(0) \end{bmatrix} = 0 \quad (2.25)$$

**S-Controllability** The descriptor linear system (2.17) is said to be *strongly controllable* or S-Controllable if it satisfies the conditions of R-Controllability and I-Controllability.

The relations between the different concepts of controllability are described by the diagram of [Duan, 2010]:

$$\text{C-Controllability} \Rightarrow \text{S-Controllability} \Leftrightarrow \text{R-Controllability} + \text{I-Controllability}$$

**C-Observability** We define observability as the ability of reconstruct the state of the system from the inputs and output measurements. Like the controllability concept, observability have multiple definitions for descriptor systems based on the ability to reconstruct the slow and the fast system (based on descriptor systems of the form (2.17)). For this section we present a descriptor system of the form

$$E\dot{x}(t) = Ax(t) \quad (2.26a)$$

$$y(t) = Cx(t) \quad (2.26b)$$

using the single value decomposition, the descriptor system (2.26) can be represented in the form

$$\dot{x}_1(t) = A_1x_1(t) \quad (2.27a)$$

$$y_1(t) = C_1x_1(t)$$

$$N\dot{x}_2(t) = x_2(t) \quad (2.27b)$$

$$y_2(t) = C_2x_2(t)$$

where  $x_1(t) \in \mathbb{R}^{n_1}$  and  $x_2(t) \in \mathbb{R}^{n_2}$ . The pair of transformation is assumed to be  $(P, Q)$  this gives us that  $P^{-1}x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ ,  $QEP = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}$ ,  $QAP = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix}$  and  $CP = [C_1 \ C_2]$ .

We say then that a regular descriptor system is *completely observable* or C-Observable if the initial condition  $x(0)$  can be uniquely determined by  $u(t)$  and  $y(t)$ . The observability of a

descriptor linear system can be determined by its observability matrices  $\mathcal{Q}_0$  for systems of the form (2.27) we have two observability matrices, one for the fast system and one the the slow one. These are defined as

$$\mathcal{Q}_0[A_1, C_1] = \begin{bmatrix} C_1 \\ C_1 A_1 \\ \vdots \\ C_1 A_1^{n-1} \end{bmatrix} \quad (2.28)$$

and

$$\mathcal{Q}_0[N, C_2] = \begin{bmatrix} C_2 \\ C_2 N \\ \vdots \\ C_2 N^{n-1} \end{bmatrix} \quad (2.29)$$

This leads to the following definition

**Definition 2.1.7** *The regular descriptor system is C-Observable if and only if both, the fast and the slow systems are C-Observable. They can be determined by*

1. *The slow system is C-Observable if and only if*

$$\text{rank}(\mathcal{Q}_0[A_1, C_1]) = n_1$$

2. *The fast system is C-Observable if and only if*

$$\text{rank}(\mathcal{Q}_0[N, C_2]) = n_2$$

**R-Observability** The system (2.26) is said to be *observable within the reachable set* or R-Observable if any state in the reachable set can be uniquely determined by  $y(t)$  and  $u(t)$ .

**I-Observability** The regular descriptor system (2.26) is *impulse observable* or I-Observable if the impulse behavior in the state response  $x(t)$  can be determined uniquely by the impulse behavior of the output and the jump behavior in the input.

**S-Observability** As for the controllability, a regular descriptor system (2.26) is *strongly observable* if it is both R-Observable and I-Observable.

**Partial Observability** Consider the following consistent linear equation system of unknowns  $x$  and  $z$ .

$$\mathcal{A}x = b \quad (2.30a)$$

$$z = \mathcal{L}x \quad (2.30b)$$

Partial observability concerns the capability to reconstruct a function of the state of a system based on the measurements of its outputs. This can be deduced based on the following lemmas.

**Lemma 1** For any two matrices  $\mathcal{A}$  and  $\mathcal{L}$ ,  $\text{Im}(\mathcal{L}) \subset \text{Im}(\mathcal{A})$  if and only if there exist a matrix  $\Phi$  such that  $\mathcal{L} = \Phi\mathcal{A}$ .

The proof of this lemma is well known in linear algebra theory.

**Lemma 2** The following statements are equivalent.

1. The functional  $z$  is uniquely determined by (2.30a)

2.  $\text{Im}(\mathcal{L}) \subset \text{Im}(\mathcal{A})$

3.  $\text{rank} \begin{bmatrix} \mathcal{L} \\ \mathcal{A} \end{bmatrix} = \text{rank} \mathcal{A}$

**Proof 1** The proof of this Lemma can be found in [Darouach, 2012]

## 2.2 LMI regions

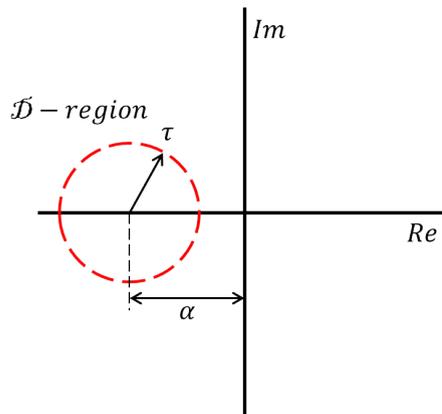
A subset of the complex plane is called an  $n$ -order LMI region if there exists a symmetric matrix  $\mathbb{M} \in \mathbb{R}^n \times n$  and a matrix  $\mathbb{N} \in \mathbb{R}^n \times n$  such that:

$$R = \{z \in \mathbb{C} : f_R(z) = \mathbb{M} + \mathbb{N}z + \mathbb{N}^T z < 0\} \quad (2.31)$$

It can be seen that the characteristic function  $f_R$  of the complex variable  $z$  takes values in the space of  $n$ -order Hermithian matrices and that the LMI region is symmetric with respect to the  $x$  axis since  $f_R^T(z) = f_R(z)$ . If all the poles of a system are in the subset  $R$ . It is said that the system is  $R$ -stable. This can be used to obtain a desired dynamical behavior of a system since we can use control feedback to place the poles of said system in a desired LMI region [Chilali and Gahinet, 1996].

### 2.2.1 $\mathcal{D}$ -stability

A system is said to be  $\mathcal{D}$ -stable if all of its eigenvalues are inside a disk region  $\mathcal{D}(\alpha, \tau)$ , centered in  $(\alpha, 0)$  and with radius  $\tau$  (see Fig. 2.3 ).

Figure 2.3:  $\mathcal{D}$ -stability region

**Lemma 3** [Zhang et al., 2016] *The eigenvalues of two given matrices  $\mathcal{A} \in \mathbb{R}^{n \times n}$ , belong to the disk region  $\mathcal{D}(\alpha, \tau)$  with center  $\alpha + 0i$  and radius  $\tau$  if and only if there exists a symmetric positive definite matrix  $\mathcal{P} \in \mathbb{R}^{n \times n}$  such that the following condition holds:*

$$\begin{bmatrix} -\mathcal{P} & \mathcal{P}\mathcal{A} - \alpha\mathcal{P} \\ * & -\tau^2\mathcal{P} \end{bmatrix} < 0 \quad (2.32)$$

## 2.3 Takagi-Sugeno systems

A Takagi-Sugeno system, also known as a Takagi-Sugeno-Kang (TSK) model, is a type of fuzzy model used in control systems. It is named after its creators, Michio Takagi, Mamoru Sugeno, and Tomomitsu Kang. A Takagi-Sugeno system is a mathematical representation of a nonlinear system using a set of fuzzy rules.

In a Takagi-Sugeno system, the overall system behavior is modeled as a collection of local linear models, where each local model corresponds to a fuzzy rule. The fuzzy rules consist of linguistic variables and fuzzy sets that describe the system's inputs and outputs. These linguistic variables are associated with membership functions that quantify the degree of membership of a given input or output to a specific fuzzy set.

Each fuzzy rule in a Takagi-Sugeno system defines a linear relationship between the system inputs and the corresponding output. These linear relationships are often defined using weighted averages, where the weights are determined by the degree of membership of the input variables in the fuzzy sets. The overall output of the Takagi-Sugeno system is then obtained by aggregating the outputs of all the fuzzy rules, typically using a weighted sum or a weighted average.

Takagi-Sugeno systems have gained popularity due to their ability to approximate nonlinear systems using a combination of local linear models. They provide a flexible and interpretable framework for modeling complex systems and designing control strategies. The parameters of

the Takagi-Sugeno model, such as the fuzzy sets and the linear coefficients, can be determined through identification techniques, including data-driven approaches and expert knowledge.

Overall, Takagi-Sugeno systems offer a powerful tool for modeling and controlling nonlinear systems, enabling the design of robust and adaptive control strategies based on fuzzy logic and approximate reasoning.

## 2.4 Functional observers

The problem of reconstructing the state of a system is one of the most important topics in control theory, since the works of D. Luenberger in 1966 where he presents an observer structure for Linear Time-Invariant (LTI) systems of the form:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \\ z(t) &= Lx(t)\end{aligned}\tag{2.33}$$

A state observer is an auxiliary dynamical systems that mirrors the behavior of a physical system. One of the most known is the Luenberger observer which has the form

$$\dot{\hat{x}}(t) = Ax(t) + Bu(t) + H(y(t) - C\hat{x}(t))$$

In order to guarantee the correct estimation of the state we need that the estimation error  $e(t) = x(t) - \hat{x}(t)$  converges asymptotically to zero, this can be reached through Lyapunov's stability with:

$$\dot{e} = (A - HC)e(t)$$

In this case  $\lim_{t \rightarrow \infty} e(t) = 0$  if the matrix  $(A - HC)$  is Hurwitz.

A functional observer estimates a function of the state vector without necessarily estimate the full state.

$$\begin{aligned}\dot{\zeta}(t) &= N\zeta(t) + Fy(t) + Hu(t) \\ \hat{z}(t) &= P\zeta(t) + Qy(t)\end{aligned}\tag{2.34}$$

The estimation error  $e(t) = \hat{z}(t) - z(t)$  converges asymptotically to zero if the matrix  $N$  in Hurwitz and there exists a matrix  $T$  such that the following conditions are satisfied:

$$\begin{aligned}NTE + FC - TA &= 0 \\ H - TB &= 0 \\ PTE + QC - L &= 0\end{aligned}$$

Functional observers present the following advantages in comparison to full order state observers

- We can reduce the order of complexity of the observer.
- We can estimate a specific number of states.
- Since control signals are often linear combinations of the states, we can estimate a control signal for the stability of the system.

### Finite-time convergence

For the case of continuous-time observers, the convergence of the estimation error to zero is always asymptotic with time. The convergence rate is exponential and can be modified by choosing the observer eigenvalues. In contrast, the observation problem in a discrete-time setting allows the choice of eigenvalues of zero and thereby a transient evolution which converges in a given number of steps, thus converging in a finite time. The guaranteed convergence time is then  $n$  times the sampling time and can be assigned by choosing the latter [Engel and Kreisselmeier, 2002]. Finite-convergence can be achieved in continuous time estimation by taking advantage of the discrete-time observers capabilities.

The following Lemma will be used later in this document.

**Lemma 4** [Zhang et al., 2016] *For two given matrices  $M_1, M_2 \in \mathbb{R}^{n \times n}$ , if  $Re(\lambda_i(M_2)) < \sigma < Re(\lambda_j(M_1)) < 0$ ,  $i, j = 1, 2, \dots, n$  for  $\sigma < 0$ , then matrix*

$$\begin{bmatrix} I & e^{M_1 h} \\ I & e^{M_2 h} \end{bmatrix}$$

*is non singular for any  $h > 0$*

## 2.5 Conclusion

Chapter 2 has laid the cornerstone for the research journey undertaken in this thesis, meticulously examining the theoretical foundations crucial to understanding functional observers and control strategies for descriptor systems. By traversing the intricacies of descriptor systems in Section 2.1, we gained insights into their structures through illustrative examples and unveiled key properties that serve as the building blocks for subsequent analyses.

The exploration of Linear Matrix Inequality (LMI) regions in Section 2.2, with a specific focus on D-stability, has provided a mathematical framework for assessing system stability. This mathematical lens is instrumental in evaluating the robustness and reliability of control strategies, laying the groundwork for the subsequent chapters.

The extension to Takagi-Sugeno systems in Section 2.3 broadens the theoretical scope, demonstrating their applicability in addressing nonlinear complexities. The inclusion of functional observers in Section 2.4 completes the theoretical framework, highlighting their pivotal role in system observation and control. This comprehensive overview equips us with a versatile set of tools essential for advancing the understanding and application of functional observers and control strategies within the realm of descriptor systems.

As we move forward in the subsequent chapters, the theoretical foundation established here will serve as a guiding beacon, enabling us to delve into more specific aspects of functional observers and control for descriptor systems. By providing this theoretical groundwork, Chapter 2 paves the way for the practical implementation and refinement of these techniques, offering valuable insights for researchers and practitioners seeking to navigate the complexities of descriptor systems in diverse real-world scenarios.



# Chapter 3

## Functional observers for LTI systems

### Contents

---

<b>3.1</b>	<b>Functional observer for descriptor systems . . . . .</b>	<b>30</b>
3.1.1	Class of systems considered . . . . .	30
3.1.2	Finite-Time Generalized Functional Observer Design . . . . .	32
3.1.3	GDO design for LTI systems . . . . .	40
<b>3.2</b>	<b>Functional observer for descriptor systems with unknown input . . . . .</b>	<b>49</b>
3.2.1	Problem statement . . . . .	50
3.2.2	Observer parameterization . . . . .	52
3.2.3	Observer design . . . . .	55
<b>3.3</b>	<b>Functional observer for T-S descriptor systems . . . . .</b>	<b>57</b>
3.3.1	Problem statement . . . . .	57
3.3.2	Observer parameterization . . . . .	59
3.3.3	Functional observer design . . . . .	61
3.3.4	Particular case: Proportional Functional observer . . . . .	62
3.3.5	Mathematical model . . . . .	63
3.3.6	Results . . . . .	64
3.3.7	Application to state and fault estimation for LTI systems . . . . .	67
<b>3.4</b>	<b>Conclusions . . . . .</b>	<b>70</b>

---

In practical applications, a myriad of control functions, such as fault diagnosis and monitoring, necessitate the manipulation of the state vector  $x(t)$ . However, obtaining the complete vector is often unfeasible due to exorbitant measurement costs or physical constraints. In such scenarios, an estimator proves invaluable in generating an estimate  $\hat{x}(t)$  of the state vector  $x(t)$ . With the increasing complexity of controlled systems, there is a growing emphasis on ensuring safety and reliability in fault detection and estimation schemes.

Faults, defined as unpredictable alterations in the system's behavior leading to performance degradation, have become a focal point in control system design. The development of effective fault-tolerant control hinges on obtaining precise estimates of both the state variables and faults. Finite-time convergence has emerged as a captivating area of study [Engel and Kreisselmeier, 2002, Zhang et al., 2016], finding applications in diverse fields such as communication systems [Perruquetti et al., 2008], mechanical systems [Davila et al., 2005], and nonlinear systems [Shen and Xia, 2008]. The significance of finite-time convergence is particularly pronounced in fault estimation, given the prevalent need for rapid response constraints in practical control systems.

## 3.1 Functional observer for descriptor systems

In this section, we introduce a finite-time convergent estimator leveraging the framework of generalized functional observers. Capitalizing on the inherent structure of functional observers, we present a reduced-order observer designed to concurrently provide finite-time estimations of both the system's state and any faults present. This innovative approach enhances the efficiency and accuracy of state and fault estimation within a defined finite time period.

### 3.1.1 Class of systems considered

Let us consider the linear continuous time-invariant descriptor system described by the following equation:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \\ z(t) &= Lx(t) \end{aligned} \tag{3.1}$$

where  $x(t) \in \mathbb{R}^n$  is the state of the system,  $u(t) \in \mathbb{R}^m$  is the input of the system,  $y(t) \in \mathbb{R}^p$  is the measured output, and  $z(t) \in \mathbb{R}^q$  is a function of the state to be estimated. Matrix  $E \in \mathbb{R}^{n \times n}$  is singular, and  $A$ ,  $B$ ,  $C$  and  $L$  are known constant matrices of appropriate dimensions. Let  $\text{rank}(E) = r < n$  and  $E^\perp \in \mathbb{R}^{(n-r) \times n}$  be a full row rank matrix such that  $E^\perp E = 0$ , where  $r_1 = n - r$ .

**Lemma 5** [Garcia and Bernussou, 1995] *The eigenvalues of a given matrix  $\mathcal{A} \in \mathbb{R}^{n \times n}$ , belong to the disk region  $\mathcal{Q}(\alpha, \tau)$  with center  $\alpha$  and radius  $\tau$  if and only if, there exists a symmetric positive definite matrix  $\mathcal{P} \in \mathbb{R}^{n \times n}$  such that the following condition holds:*

$$\begin{bmatrix} -\mathcal{P}^{-1} & \frac{1}{\tau}(\mathcal{A} - \alpha I) \\ (*) & -\mathcal{P} \end{bmatrix} < 0 \tag{3.2}$$

By pre-multiplying and post-multiplying this inequality by  $\begin{bmatrix} \mathcal{P} & 0 \\ 0 & I \end{bmatrix}$ , we obtain the following equivalent LMI.

$$\begin{bmatrix} -\mathcal{P} & \frac{1}{\tau}\mathcal{P}(\mathcal{A} - \alpha I) \\ (*) & -\mathcal{P} \end{bmatrix} < 0 \tag{3.3}$$

By applying the result of Lemma 5 to two matrices  $\mathcal{A}_1$  and  $\mathcal{A}_2$  we obtain the result of the following lemma.

**Lemma 6** [Engel and Kreisselmeier, 2002] For two matrices  $\mathcal{A}_1, \mathcal{A}_2 \in \mathbb{R}^{n \times n}$ , if  $Re(\lambda_i(\mathcal{A}_2)) < \sigma < Re(\lambda_j(\mathcal{A}_1)) < 0$ ,  $i, j = 1, 2, \dots, n$  for  $\sigma < 0$ , then the matrix

$$\begin{bmatrix} I & e^{\mathcal{A}_1 h} \\ I & e^{\mathcal{A}_2 h} \end{bmatrix} \quad (3.4)$$

is nonsingular for any  $h > 0$ ,  $\lambda_{i,j}(\mathcal{A}_{1,2})$  denotes the eigenvalues of matrix  $\mathcal{A}_{1,2}$ , respectively.

The following Lemma allows to parameterize all the solutions of an LMI.

**Lemma 7** [Skelton et al., 1997] Let matrices  $\mathcal{B}, \mathcal{C}, \mathcal{Q} = \mathcal{Q}^T$  be given, then the following statements are equivalent:

1. There exists a matrix  $\mathcal{X}$  satisfying

$$\mathcal{B}\mathcal{X}\mathcal{C} + (\mathcal{B}\mathcal{X}\mathcal{C})^T + \mathcal{Q} < 0$$

2. The following conditions hold

$$\mathcal{B}^\perp \mathcal{Q} \mathcal{B}^{\perp T} < 0 \text{ or } \mathcal{B}\mathcal{B}^T > 0$$

$$\mathcal{C}^{T\perp} \mathcal{Q} \mathcal{C}^{T\perp T} < 0 \text{ or } \mathcal{C}^T \mathcal{C} > 0$$

where  $^T$  denotes the transpose of the matrix. Suppose the above statements hold. Let  $r_b$  and  $r_c$  be the ranks of  $\mathcal{B}$  and  $\mathcal{C}$ , respectively, and  $(\mathcal{B}_l, \mathcal{B}_r)$  and  $(\mathcal{C}_l, \mathcal{C}_r)$  be any full rank factors of  $\mathcal{B}$  and  $\mathcal{C}$ , i.e.,  $\mathcal{B} = \mathcal{B}_l \mathcal{B}_r$ ,  $\mathcal{C} = \mathcal{C}_l \mathcal{C}_r$ . Then all matrices  $\mathcal{X}$  in statement (1) of Lemma 7 are given by

$$\mathcal{X} = \mathcal{B}_r^+ \mathcal{K} \mathcal{C}_l^+ + \mathcal{Z} - \mathcal{B}_r^+ \mathcal{B}_r \mathcal{Z} \mathcal{C}_l \mathcal{C}_l^+$$

where  $\mathcal{Z}$  is an arbitrary matrix, the symbol  $^+$  denotes the generalized inverse.

$$\mathcal{K} \triangleq -\mathcal{R}^{-1} \mathcal{B}_l^T \Theta \mathcal{C}_r^T (\mathcal{C}_r \Theta \mathcal{C}_r^T)^{-1} + \mathcal{S}^{1/2} \mathcal{L} (\mathcal{C}_r \Theta \mathcal{C}_r^T)^{-1/2}$$

$$\mathcal{S} \triangleq \mathcal{R}^{-1} - \mathcal{R}^{-1} \mathcal{B}_l^T [\Theta - \Theta \mathcal{C}_r^T (\mathcal{C}_r \Theta \mathcal{C}_r^T)^{-1} \mathcal{C}_r \Theta] \mathcal{B}_l \mathcal{R}^{-1}$$

where  $\mathcal{L}$  is an arbitrary matrix such that  $\|\mathcal{L}\| < 1$  and  $\mathcal{R}$  is an arbitrary positive definite matrix such that

$$\Theta \triangleq (\mathcal{B}_l \mathcal{R}^{-1} \mathcal{B}_l^T - \mathcal{Q})^{-1} > 0$$

Let us consider a class of descriptor linear time-invariant systems of the form (3.1). In the sequel we assume that system (3.1) is regular and the triplet  $(E, A, C)$  is partially impulse observable with respect to matrix  $L$  which is equivalent to the following assumption .

**Assumption 1** [Darouach, 2012] *Functional observability allows us to verify if the linear function of the states can be estimated, from the information available from the input and output, this is satisfied if the following condition holds.*

$$\text{rank} \begin{bmatrix} L \\ E \\ E^\perp A \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} E \\ E^\perp A \\ C \end{bmatrix}$$

The reconstruction of the function  $z(t)$  in finite-time can be obtained by using two functional generalized dynamical observers of the form:

$$\begin{aligned} \dot{\zeta}_i(t) &= N_i \zeta_i(t) + J_i v_i(t) + F_i \begin{bmatrix} -E^\perp B u(t) \\ y(t) \end{bmatrix} + H_i u(t) \\ \dot{v}_i(t) &= S_i \zeta_i(t) + G_i v_i(t) + M_i \begin{bmatrix} -E^\perp B u(t) \\ y(t) \end{bmatrix} \\ \hat{z}_i(t) &= P_i \zeta_i(t) + Q_i \begin{bmatrix} -E^\perp B u(t) \\ y(t) \end{bmatrix} \end{aligned} \quad (3.5)$$

for  $i = 1, 2$ , where  $\zeta_i(t) \in \mathbb{R}^{q_0}$  are the states of the observers,  $v_i(t) \in \mathbb{R}^{q_1}$  are auxiliary vectors and  $\hat{z}_i(t) \in \mathbb{R}^q$  are the estimate of  $z(t)$ .  $N_i, J_i, F_i, H_i, S_i, G_i, M_i, P_i$  and  $Q_i$  are constant matrices of appropriate dimensions to be determined. Both observers have the same structure but different parameter matrices. Before continuing, we make the following remarks.

**Remark 1** *For standard systems where  $E = I$  Assumption 1 is always satisfied. Also when  $L = I$ , the case of the full-order observer is presented, and Assumption 1 is reduced to  $\text{rank} \begin{bmatrix} E \\ E^\perp A \\ C \end{bmatrix} = n$  or  $\text{rank} \begin{bmatrix} E & A \\ 0 & C \\ 0 & E \end{bmatrix} = n + \text{rank}(E)$ , according to Lemma 3, which is the impulse observability of the triplet  $(E, A, C)$ .*

**Remark 2** *The form of the observer (3.5) is general and the Proportional Observer (PO) and the Proportional Integral observer (PIO) can be obtained as particular cases of (3.5). In fact.*

- For  $J_i = 0, S_i = 0, M_i = 0$  and  $G_i = 0$ , we obtain the P Observer.
- For  $S_i = 0$  the observer becomes a PI Observer.

*The form of the observer given in (3.5) presents an additional parameters which can introduce some slack variables, these variables can be used to realize other performances of the observer.*

### 3.1.2 Finite-Time Generalized Functional Observer Design

Consider parameter matrices  $T_i \in \mathbb{R}^{q_0 \times n}$  and define the transformed error vectors

$$\varepsilon_i(t) = \zeta_i(t) - T_i E x(t) \quad (3.6)$$

Since  $E^\perp E = 0$ , it can be seen from (3.1) that,  $E^\perp Ax(t) = -E^\perp Bu(t)$ , such that the derivative of (3.6) is given by

$$\dot{\varepsilon}_i(t) = N_i \varepsilon_i(t) + \left( N_i T_i E + F_i \begin{bmatrix} E^\perp A \\ C \end{bmatrix} - T_i A \right) x(t) + (H_i - T_i B)u(t) + J_i v_i(t) \quad (3.7)$$

from (3.5),  $\dot{v}_i(t)$  can be rewritten as

$$\dot{v}_i(t) = S_i \varepsilon_i(t) + G_i v_i(t) + \left( M_i \begin{bmatrix} E^\perp A \\ C \end{bmatrix} + S_i T_i E \right) x(t) \quad (3.8)$$

If the following conditions are satisfied

**a.**  $N_i T_i E + F_i \begin{bmatrix} E^\perp A \\ C \end{bmatrix} - T_i A = 0$

**b.**  $H_i = T_i B$

**c.**  $S_i T_i E + M_i \begin{bmatrix} E^\perp A \\ C \end{bmatrix} = 0$ , for  $i=1,2$

**d.**  $P_i T_i E + Q_i \begin{bmatrix} E^\perp A \\ C \end{bmatrix} = L$ , for  $i=1,2$

then (3.7) and (3.8) can be written as

$$\dot{\varepsilon}_i(t) = N_i \varepsilon_i(t) + J_i v_i(t) \quad (3.9)$$

$$\dot{v}_i(t) = S_i \varepsilon_i(t) + G_i v_i(t) \quad (3.10)$$

By defining an augmented state vector  $\vartheta_i(t) = \begin{bmatrix} \varepsilon_i(t) \\ v_i(t) \end{bmatrix}$ , (3.9) and (3.10) can be rewritten as:

$$\dot{\vartheta}_i(t) = \underbrace{\begin{bmatrix} N_i & J_i \\ S_i & G_i \end{bmatrix}}_{\mathbb{A}_i} \vartheta_i(t) \quad (3.11)$$

Now, from (3.5) we obtain

$$\hat{z}_i(t) = P_i \varepsilon_i(t) + \left( P_i T_i E + Q_i \begin{bmatrix} E^\perp A \\ C \end{bmatrix} \right) x(t) \quad (3.12)$$

If condition d. is satisfied, then

$$e_i(t) = P_i \varepsilon_i(t) \quad (3.13)$$

where  $e_i(t) = \hat{z}_i(t) - z(t)$ .

It can be seen that, if matrices  $\mathbb{A}_i$  from (3.11) are Hurwitz, then  $\lim_{t \rightarrow \infty} \vartheta_i(t) = 0$  and  $\lim_{t \rightarrow \infty} e_i(t) = 0$ , for  $i = 1, 2$ .

### Observer parameterization

From conditions **a.** to **d.**, it can be seen that the design problem of the two observers is reduced to find the matrices  $N_i, J_i, F_i, H_i, S_i, G_i, M_i, P_i, Q_i$  and  $T_i$  such that  $\mathbb{A}_i$  is Hurwitz.

Define matrix  $\Gamma = \begin{bmatrix} E \\ E^\perp A \\ C \end{bmatrix}$  and let  $R_i \in \mathbb{R}^{q_0 \times n}$ , for  $i = 1, 2$ , be a full row rank matrix such that

$\text{rank} \begin{bmatrix} R_i \\ \Gamma \end{bmatrix} = \text{rank}(\Gamma)$ . Also let us define the matrix  $\Sigma_i = \begin{bmatrix} R_i \\ E^\perp A \\ C \end{bmatrix}$ . In this case there always

exists two matrices  $T_i$  and  $K_i$  such that

$$T_i E + K_i \begin{bmatrix} E^\perp A \\ C \end{bmatrix} = R_i \quad (3.14)$$

which can be rewritten as

$$[T_i \quad K_i] \Gamma = R_i \quad (3.15)$$

the general solution of (3.15) is given by

$$[T_i \quad K_i] = R_i \Gamma^+ - Z_i (I - \Gamma \Gamma^+) \quad (3.16)$$

where  $\Gamma^+$  is the generalized inverse of  $\Gamma$ .

Equation (3.16) leads to

$$T_i = T_{i1} - Z_i T_2 \quad (3.17)$$

$$K_i = K_{i1} - Z_i K_2 \quad (3.18)$$

where  $Z_i$  is a matrix of appropriate dimensions with arbitrary elements, and  $T_{i1} = R_i \Gamma^+ \begin{bmatrix} I \\ 0 \end{bmatrix}$ ,

$T_2 = (I - \Gamma \Gamma^+) \begin{bmatrix} I \\ 0 \end{bmatrix}$ ,  $K_{i1} = R_i \Gamma^+ \begin{bmatrix} 0 \\ I \end{bmatrix}$  and  $K_2 = (I - \Gamma \Gamma^+) \begin{bmatrix} 0 \\ I \end{bmatrix}$ .

From condition **a.** and (3.14), we have:

$$N_i \left( R_i - K_i \begin{bmatrix} E^\perp A \\ C \end{bmatrix} \right) + F_i \begin{bmatrix} E^\perp A \\ C \end{bmatrix} = T_i A \quad (3.19)$$

which can be written as

$$[N_i \quad \tilde{K}_i] \Sigma_i = T_i A \quad (3.20)$$

where  $\tilde{K}_i = F_i - N_i K_i$ . If  $\text{rank} \begin{bmatrix} \Sigma_i \\ T_i A \end{bmatrix} = \text{rank}(\Sigma_i)$ , the general solution to (3.20) is

$$[N_i \quad \tilde{K}_i] = T_i A \Sigma_i^+ - Y_{i1} (I - \Sigma_i \Sigma_i^+) \quad (3.21)$$

where  $Y_{i1}$  is a matrix of appropriate dimensions with arbitrary elements.

By replacing (3.17) into (3.21), we obtain

$$N_i = N_{i1} - Z_i N_{i2} - Y_{i1} N_{i3} \quad (3.22)$$

$$\tilde{K}_i = \tilde{K}_{i1} - Z_i \tilde{K}_{i2} - Y_{i1} \tilde{K}_{i3} \quad (3.23)$$

where  $N_{i1} = T_{i1}A\Sigma_i^+ \begin{bmatrix} I \\ 0 \end{bmatrix}$ ,  $N_{i2} = T_{i2}A\Sigma_i \begin{bmatrix} I \\ 0 \end{bmatrix}$ ,  $N_{i3} = (I - \Sigma_i\Sigma_i^+) \begin{bmatrix} I \\ 0 \end{bmatrix}$ ,  $\tilde{K}_{i1} = T_{i1}A\Sigma_i^+ \begin{bmatrix} 0 \\ I \end{bmatrix}$ ,  $\tilde{K}_{i2} = T_{i2}A\Sigma_i^+ \begin{bmatrix} 0 \\ I \end{bmatrix}$  and  $\tilde{K}_{i3} = (I - \Sigma_i\Sigma_i^+) \begin{bmatrix} 0 \\ I \end{bmatrix}$ .

Since  $F_i = \tilde{K}_i + N_iK_i$ , we have

$$\begin{aligned} F_i &= \tilde{K}_{i1} + N_{i1}K_i - Z_i(\tilde{K}_{i2} - N_{i2}K_i) - Y_{i1}(\tilde{K}_{i3} - N_{i3}K_i) \\ F_i &= F_{i1} - Z_iF_{i2} - Y_{i1}F_{i3} \end{aligned} \quad (3.24)$$

where  $F_{i1} = T_{i1}A\Sigma_i^+ \begin{bmatrix} K_i \\ I \end{bmatrix}$ ,  $F_{i2} = T_{i2}A\Sigma_i^+ \begin{bmatrix} K_i \\ I \end{bmatrix}$  and  $F_{i3} = (I - \Sigma_i\Sigma_i^+) \begin{bmatrix} K_i \\ I \end{bmatrix}$ .

From (3.14) we have

$$\begin{bmatrix} T_iE \\ E^\perp A \\ C \end{bmatrix} = \begin{bmatrix} I & -K_i \\ 0 & I \end{bmatrix} \Sigma_i \quad (3.25)$$

Conditions **c.** and **d.** can be written as

$$\begin{bmatrix} S_i & M_i \\ P_i & Q_i \end{bmatrix} \begin{bmatrix} T_iE \\ E^\perp A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ L \end{bmatrix} \quad (3.26)$$

If we substitute (3.25) into (3.26), we obtain

$$\begin{bmatrix} S_i & M_i \\ P_i & Q_i \end{bmatrix} \begin{bmatrix} I & -K_i \\ 0 & I \end{bmatrix} \Sigma_i = \begin{bmatrix} 0 \\ L \end{bmatrix} \quad (3.27)$$

By using the fact that  $\begin{bmatrix} I & -K_i \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & K_i \\ 0 & I \end{bmatrix}$  and considering that  $\text{rank} \begin{bmatrix} \Sigma_i \\ L \end{bmatrix} = \text{rank}(\Sigma_i)$ , the general solution to (3.27) is given by

$$\begin{bmatrix} S_i & M_i \\ P_i & Q_i \end{bmatrix} = \left( \begin{bmatrix} 0 \\ L \end{bmatrix} \Sigma_i^+ - Y_i (I - \Sigma_i\Sigma_i^+) \right) \begin{bmatrix} I & K_i \\ 0 & I \end{bmatrix} \quad (3.28)$$

where  $Y_i = \begin{bmatrix} Y_{i2} \\ Y_{i3} \end{bmatrix}$  is a matrix of appropriate dimensions with arbitrary elements. In this case, the particular solutions for  $S_i$ ,  $M_i$ ,  $P_i$  and  $Q_i$  are given by

$$S_i = -Y_{i2}N_{i3} \quad (3.29)$$

$$M_i = -Y_{i2}F_{i3} \quad (3.30)$$

$$P_i = P_{i1} - Y_{i3}N_{i3} \quad (3.31)$$

$$Q_i = Q_{i1} - Y_{i3}F_{i3} \quad (3.32)$$

where  $P_{i1} = L\Sigma_i^+ \begin{bmatrix} I \\ 0 \end{bmatrix}$  and  $Q_{i1} = L\Sigma_i^+ \begin{bmatrix} K_i \\ I \end{bmatrix}$ .

We can see that the design is independent of the parameter matrices  $Y_{i3}$ , which can take an

arbitrarily value. For simplicity we can take  $Y_{i3} = 0$ .

Now, by using (3.22) and (3.29), the error dynamics (3.11) can be rewritten as

$$\dot{\vartheta}_i(t) = (\mathbb{A}_{i1} - \mathbb{Y}_i \mathbb{A}_{i2}) \vartheta_i(t) \quad (3.33)$$

where  $\mathbb{A}_{i1} = \begin{bmatrix} N_{i1} - Z_i N_{i2} & 0 \\ 0 & 0 \end{bmatrix}$ ,  $\mathbb{Y} = \begin{bmatrix} Y_{i1} & J_i \\ Y_{i2} & G_i \end{bmatrix}$  and  $\mathbb{A}_{i2} = \begin{bmatrix} N_{i3} & 0 \\ 0 & -I \end{bmatrix}$ .

The problem is now reduced to find matrices  $\mathbb{Y}_i$  and  $Z_i$  such that matrices  $\mathbb{A}_i$  are Hurwitz. This can be reached by using any pole placement method or a linear matrix inequality (LMI) approach.

Now, the necessary and sufficient condition for the existence of matrices  $\mathbb{Y}_i$  such that matrices  $\mathbb{A}_i$  are Hurwitz is that the pair  $(\mathbb{A}_{i2}, \mathbb{A}_{i1})$  must be detectable.

The following Lemma gives the necessary and sufficient conditions for the detectability of the pair  $(\mathbb{A}_{i2}, \mathbb{A}_{i1})$ .

**Lemma 8** *The following statements are equivalent*

1. The pair  $(\mathbb{A}_{i2}, \mathbb{A}_{i1})$  is detectable.
2.  $\text{rank} \begin{bmatrix} \lambda I - N_{i1} + Z_i N_{i2} \\ N_{i3} \end{bmatrix} = q_0, \forall \lambda \in \mathbb{C}, \text{Re}(\lambda) \geq 0$

**Proof 2** *The pair  $(\mathbb{A}_{i1}, \mathbb{A}_{i2})$  is detectable if and only if*

$$\text{rank} \begin{bmatrix} \lambda I - \mathbb{A}_{i1} \\ \mathbb{A}_{i2} \end{bmatrix} = q_0 + q_1, \forall \lambda \in \mathbb{C}, \text{Re}(\lambda) \geq 0$$

or equivalently

$$\text{rank} \begin{bmatrix} \lambda I - N_{i1} + Z_i N_{i2} & 0 \\ 0 & \lambda I \\ N_{i3} & 0 \\ 0 & -I \end{bmatrix} = q_0 + q_1, \forall \lambda \in \mathbb{C}, \text{Re}(\lambda) \geq 0 \quad (3.34)$$

**Remark 3** *From the result of the above lemma it is easy to show that a sufficient condition for matrix  $\mathbb{A}_i$  to be Hurwitz is*

$$\text{rank} \begin{bmatrix} \lambda I - N_{i1} \\ N_{i2} \\ N_{i3} \end{bmatrix} = q_0, \forall \lambda \in \mathbb{C}, \text{Re}(\lambda) \geq 0. \quad (3.35)$$

### Finite-time observer design

From the precedent results, the design of a generalized finite time observer is reduced to determine the parameters  $\mathbb{Y}_i$  and  $Z_i$  such that the eigenvalues of matrices  $\mathbb{A}_i$  are inside two disks regions  $\mathcal{Q}_1(\alpha_1, \tau_1)$  and  $\mathcal{Q}_2(\alpha_2, \tau_2)$  by using Lemma 5. The following theorem gives the sufficient conditions in LMIs formulation to realize this objective.

**Theorem 1** Under assumption 1 and condition (3.35), let  $\mathcal{Q}_i(\alpha_i, \tau_i), i = 1, 2$  be two disks region, then there exist parameters matrices  $\mathbb{Y}_i$  and  $Z_i$  such that the eigenvalues of matrices  $\mathbb{A}_i$  are inside these regions, if there exists two symmetric positive definite matrices  $X_i = \begin{bmatrix} X_{i1} & X_{i1} \\ X_{i1} & X_{i2} \end{bmatrix} \in \mathbb{R}^{(q_0+q_1) \times (q_0+q_1)}$  such that the following LMIs are satisfied:

$$\begin{bmatrix} -X_i & \tilde{\mathbb{A}}_i \\ (*) & -\mathbb{A}_{i2}^{T\perp} X_i \mathbb{A}_{i2}^{T\perp T} \end{bmatrix} < 0 \quad (3.36)$$

where  $\tilde{\mathbb{A}}_i = \frac{1}{\tau_i} \begin{bmatrix} (X_{i1} N_{i1} - W_i N_{i2} - \alpha_i X_{i1}) N_{i3}^{T\perp T} \\ (X_{i1} N_{i1} - W_i N_{i2} - \alpha_i X_{i1} i) N_{i3}^{T\perp T} \end{bmatrix}$ ,  $W_i = X_{i1} Z_i$  and matrices  $\mathbb{Y}_i$  can be parameterized as it follows:

$$\mathbb{Y}_i = -\tau_i (\mathcal{B}_{ir}^+ \mathcal{K}_i \mathcal{C}_{il}^+ + \mathcal{Z}_i - \mathcal{B}_{ir}^+ \mathcal{B}_{ir} \mathcal{Z}_i \mathcal{C}_{il} \mathcal{C}_{il}^+) \quad (3.37)$$

where

$$\mathcal{K}_i = -\mathcal{R}_i^{-1} \mathcal{B}_{il}^T \Theta_i \mathcal{C}_{ir}^T (\mathcal{C}_{ir} \Theta_i \mathcal{C}_{ir}^T)^{-1} + \mathcal{S}_i^{1/2} \mathcal{L}_i (\mathcal{C}_{ir} \Theta_i \mathcal{C}_{ir}^T)^{-1/2} \quad (3.38)$$

$$\mathcal{S}_i = \mathcal{R}_i^{-1} - \mathcal{R}_i^{-1} \mathcal{B}_{il}^T [\Theta_i - \Theta_i \mathcal{C}_{ir}^T (\mathcal{C}_{ir} \Theta_i \mathcal{C}_{ir}^T)^{-1} \mathcal{C}_{ir} \Theta_i] \mathcal{B}_{il} \mathcal{R}_i^{-1} \quad (3.39)$$

and where  $\mathcal{L}_i$  is an arbitrary matrix such that  $\|\mathcal{L}_i\| < 1$  and  $\mathcal{R}_i$  is an arbitrary positive definite matrix such that

$$\Theta_i = (\mathcal{B}_{il} \mathcal{R}_i^{-1} \mathcal{B}_{il}^T - \mathcal{Q}_i)^{-1} > 0$$

for  $i = 1, 2$ .

**Proof 3** Under Assumption 1 and Condition (3.35), from Lemma 5 the eigenvalues of matrices  $\mathbb{A}_i$  are inside the two disks region  $\mathcal{Q}_i(\alpha_i, \tau_i), i = 1, 2$ , if the following LMIs are satisfied

$$\begin{bmatrix} -X_i & \frac{1}{\tau_i} (X_i \mathbb{A}_i - \alpha_i X_i) \\ (*) & -X_i \end{bmatrix} < 0 \quad (3.40)$$

where the symbol  $(*)$  denotes the transpose elements in the symmetric positions. Also (3.40) can be written as

$$\mathcal{B}_i \mathcal{X}_i \mathcal{C}_i + (\mathcal{B}_i \mathcal{X}_i \mathcal{C}_i)^T + \mathcal{Q}_i < 0 \quad (3.41)$$

where  $\mathcal{B}_i = \begin{bmatrix} X_i \\ 0 \end{bmatrix}$ ,  $\mathcal{X}_i = -\frac{1}{\tau_i} \mathbb{Y}_i$ ,  $\mathcal{C}_i = [0 \quad \mathbb{A}_{i2}]$ , and  $\mathcal{Q}_i = \begin{bmatrix} -X_i & \frac{1}{\tau_i} (X_i \mathbb{A}_{i1} - \alpha_i X_i) \\ (*) & -X_i \end{bmatrix}$ . From Lemma 7, the solvability conditions of (3.41) are

$$\mathcal{B}_i^\perp \mathcal{Q}_i \mathcal{B}_i^{\perp T} < 0 \quad (3.42)$$

$$\mathcal{C}_i^{T\perp} \mathcal{Q}_i \mathcal{C}_i^{T\perp T} < 0 \quad (3.43)$$

since  $\mathcal{B}_i^\perp = [0 \quad I]$ , inequality (3.42) becomes  $X_i > 0$  and considering  $\mathcal{C}_i^{T\perp} = \begin{bmatrix} I & 0 \\ 0 & \mathbb{A}_{i2}^{T\perp} \end{bmatrix}$  we obtain

$$\begin{bmatrix} -X_i & \frac{1}{\tau_i} (X_i \mathbb{A}_{i1} - \alpha_i X_i) \mathbb{A}_{i2}^{T\perp T} \\ (*) & -\mathbb{A}_{i2}^{T\perp} X_i \mathbb{A}_{i2}^{T\perp T} \end{bmatrix} < 0 \quad (3.44)$$

By developing the expression  $X_i \mathbb{A}_{i1} - \alpha_i X_i$ , we obtain

$$X_i \mathbb{A}_{i1} - \alpha_i X_i = \begin{bmatrix} X_{i1} N_{i1} - W_i N_{i2} - \alpha_i X_{i1} & -\alpha_i X_{i1} \\ X_{i1} N_{i1} - W_i N_{i2} - \alpha_i X_{i1} & -\alpha_i X_{i2} \end{bmatrix} \quad (3.45)$$

where  $W_i = X_{i1} Z_i$ . Since,  $\mathbb{A}_{i2}^{T\perp} = [N_{i3}^{T\perp} \ 0]$ , then  $(X_i \mathbb{A}_{i1} - \alpha_i X_i) \mathbb{A}_{i2}^{T\perp T}$  can be written as

$$(X_i \mathbb{A}_{i1} - \alpha_i X_i) \mathbb{A}_{i2}^{T\perp T} = \begin{bmatrix} (X_{i1} N_{i1} - W_i N_{i2} - \alpha_i X_{i1}) N_{i3}^{T\perp T} \\ (X_{i1} N_{i1} - W_i N_{i2} - \alpha_i X_{i1}) N_{i3}^{T\perp T} \end{bmatrix}$$

Let  $\tilde{\mathbb{A}}_i = \frac{1}{\tau_i} \begin{bmatrix} (X_{i1} N_{i1} - W_i N_{i2} - \alpha_i X_{i1}) N_{i3}^{T\perp T} \\ (X_{i1} N_{i1} - W_i N_{i2} - \alpha_i X_{i1}) N_{i3}^{T\perp T} \end{bmatrix}$ , then (3.44) can be written as

$$\begin{bmatrix} -X_i & \tilde{\mathbb{A}}_i \\ \tilde{\mathbb{A}}_i^T & -\mathbb{A}_{i2}^{T\perp} X_i \mathbb{A}_{i2}^{T\perp T} \end{bmatrix} < 0 \quad (3.46)$$

By using Lemma 7 we obtain the parameterization of matrix  $\mathbb{Y}_i$  as in (3.37) which completes the proof.

### Finite-Time estimator

Consider the estimation error  $e_i(t) = \hat{z}_i(t) - z(t)$  which can be written as  $e_i(t) = P_i \varepsilon_i(t)$ . By using Eq. (3.11), the estimation error becomes

$$e_i(t) = \bar{P}_i \vartheta_i(t), \text{ for } i = 1, 2 \quad (3.47)$$

where  $\bar{P}_i = [P_i \ 0]$ .

We can now express the following equations

$$\begin{aligned} \hat{z}_1(t) &= \bar{P}_1 \vartheta_1(t) + z(t) \\ \hat{z}_2(t) &= \bar{P}_2 \vartheta_2(t) + z(t) \\ \hat{z}_1(t-h) &= \bar{P}_1 \vartheta_1(t-h) + z(t-h) \\ \hat{z}_2(t-h) &= \bar{P}_2 \vartheta_2(t-h) + z(t-h) \end{aligned} \quad (3.48)$$

By using (3.11), we have

$$\vartheta_i(t) = e^{\mathbb{A}_i(t-t_0)} \vartheta_i(t_0) \quad (3.49)$$

for  $t_0 = t - h$ , with  $h$  being a scalar positive we obtain

$$\vartheta_i(t) = e^{\mathbb{A}_i h} \vartheta_i(t-h) \quad (3.50)$$

By replacing (3.50) into (3.48), we obtain

$$\begin{aligned} \hat{z}_1(t) &= \bar{P}_1 \vartheta_1(t) + z(t) \\ \hat{z}_2(t) &= \bar{P}_2 \vartheta_2(t) + z(t) \\ \hat{z}_1(t-h) &= \bar{P}_1 e^{-\mathbb{A}_1 h} \vartheta_1(t) + z(t-h) \\ \hat{z}_2(t-h) &= \bar{P}_2 e^{-\mathbb{A}_2 h} \vartheta_2(t) + z(t-h) \end{aligned} \quad (3.51)$$

Now, consider the vectors  $\hat{z}(t) = [\hat{z}_1(t) \ \hat{z}_2(t)]^T$ , and  $\vartheta(t) = [\vartheta_1(t) \ \vartheta_2(t)]^T$ , and matrices  $\bar{P} = \begin{bmatrix} \bar{P}_1 & 0 \\ 0 & \bar{P}_2 \end{bmatrix}$ ,  $e^{\mathbb{A}h} = \begin{bmatrix} e^{\mathbb{A}_1 h} & 0 \\ 0 & e^{\mathbb{A}_2 h} \end{bmatrix}$ , and  $\Psi = \begin{bmatrix} I \\ I \end{bmatrix}$ , and replace them in (3.51)

$$\hat{z}(t) = \bar{P}\vartheta(t) + \Psi z(t) \quad (3.52)$$

$$\hat{z}(t-h) = \bar{P}e^{-\mathbb{A}h}\vartheta(t) + \Psi z(t-h) \quad (3.53)$$

by replacing (3.52) in (3.51), we obtain

$$\hat{z}(t-h) = \bar{P}e^{-\mathbb{A}h}\bar{P}^+ (\hat{z}(t) - \Psi z(t)) + \Psi z(t-h) \quad (3.54)$$

which can be rewritten as

$$\hat{z}(t-h) - \Psi z(t-h) = \bar{Q} (\hat{z}(t) - \Psi z(t)) \quad (3.55)$$

where  $\bar{Q} = \bar{P}e^{-\mathbb{A}h}\bar{P}^+$ , and can be expressed as

$$[\bar{Q}\Psi \ \Psi] \begin{bmatrix} z(t) \\ -\hat{z}(t-h) \end{bmatrix} = \bar{Q}\hat{z}(t) - \hat{z}(t-h) \quad (3.56)$$

Then,  $z(t)$  can be obtained by

$$\hat{z}(t) = [I \ 0] [\bar{Q}\Psi \ \Psi]^{-1} (\bar{Q}\hat{z}(t) - \hat{z}(t-h)) \quad (3.57)$$

The existence of a finite-time convergence observer can be determined by matrix  $[\bar{Q}\Psi \ \Psi]^{-1}$ , in this case, this matrix exists if

$$\begin{aligned} \det [\bar{Q}\Psi \ \Psi] &\neq 0 \\ \det \begin{bmatrix} \bar{Q}_1 & I \\ \bar{Q}_2 & I \end{bmatrix} &\neq 0 \\ \det(\bar{Q}_1 - \bar{Q}_2) &\neq 0 \end{aligned} \quad (3.58)$$

**Remark 4** From (3.60), in the case of  $P_1 = P_2$ , the necessary condition is reduced to  $\lambda(\mathbb{A}_1) < \rho < \lambda(\mathbb{A}_2)$

### Numerical example

The following example is given to illustrate the Functional Finite-Time observer performance and its capability to estimate a control law to stabilize a Descriptor Linear Time-Invariant system. Let us consider the following Descriptor system of the form (3.1) with:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} -1 & 3 & 0 & 1 \\ 1 & -7 & 2.5 & 3 \\ 3 & -1 & 1 & 0 \\ 2 & -3.1 & -2.2 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and } C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

as we can see,  $\lambda(E, A) = \begin{bmatrix} 3.06 \\ -4.03 - 6.0i \\ -4.03 + 6.0i \end{bmatrix}$ , this makes the system unstable. The Finite-time Functional observer is designed to stabilize the system. Matrix  $L$  is obtained by a simple pole placement method. In this case we choose  $L = \begin{bmatrix} -30.09 & -8.39 & 67.22 & -1.96 \\ 12.83 & -6.5 & -47.24 & 24.11 \end{bmatrix}$  then the poles of the system are  $\lambda(E, A - BL) = \begin{bmatrix} -5.02 \\ -10.84 \\ -274.643 \end{bmatrix}$ . The performance of the Observer is tested in Simulation 1.

### Simulation 1 *Finite-time control law estimation*

The objective of this simulation is to estimate a control law for an unstable system of the form (3.1). The estimated control law is feeded to the system (see Figure 3.1). In this case, the role of the controller is satisfied by the functional observer, and the reference is set to 0.

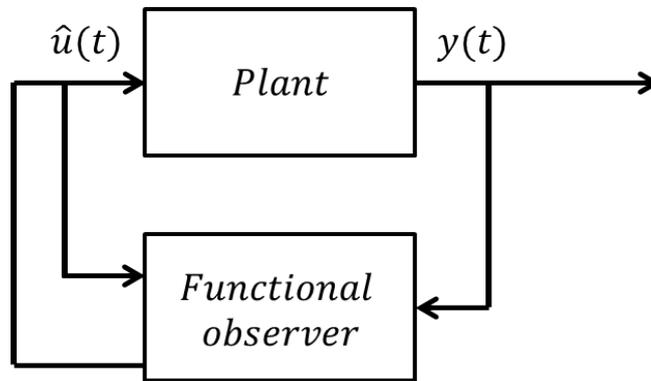


Figure 3.1: Block diagram of the control system

**Settings.** The disks regions  $\mathcal{Q}(\alpha_i, \tau_i)$  are chosen as  $\alpha_1 = -5$ ,  $\alpha_2 = -2$ ,  $\tau_1 = \tau_2 = 0.5$ , with a convergence time of 0.5s and initial condition  $x(0) = [-0.5 \ 1 \ 2.5 \ -1]^T$ . The measurable outputs are  $x_1(t)$ ,  $x_2(t)$  and  $x_4(t)$ . The results are shown in Figure 3.2 and the control law of system is shown in Figure 3.3

It can be seen in Figures 3.2 and 3.3 that the observer is capable to estimate a control law that stabilizes the system.

### 3.1.3 GDO design for LTI systems

The main contribution of the work is to propose a generalized asymptotic functional observer which extends the results presented in [Darouach, 2012]. Then a combination of two generalized functional observers is used to design a generalized finite-time functional observer. The obtained results are then applied to estimate a linear function of the state in a finite-time.

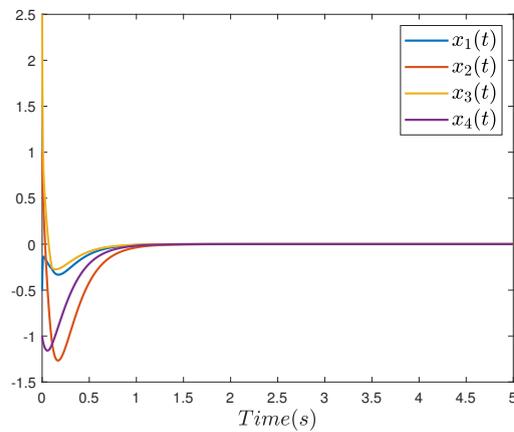


Figure 3.2: States of the LTI descriptor system

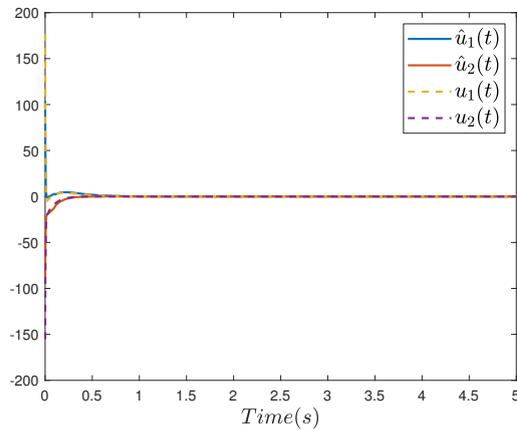


Figure 3.3: Estimated control law for the LTI descriptor system

So, a finite-time convergent estimator based on the generalized functional observer is given. We take the advantage of the generalized functional observer structure, to propose an observer capable to give a determined finite-time convergence functional estimation. A numerical example is given to illustrate our approach.

## Preliminaries

In this section we present the preliminaries that are used in the sequel of this work.

Let us consider the linear continuous time-invariant system described by the following equation:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) \\
z(t) &= Lx(t)
\end{aligned} \tag{3.59}$$

where  $x(t) \in \mathbb{R}^n$  is the state of the system,  $u(t) \in \mathbb{R}^m$  is the input of the system,  $y(t) \in \mathbb{R}^p$  is the measured output, and  $z(t) \in \mathbb{R}^q$  is a function of the state to be estimated. Matrices  $A$ ,  $B$ ,  $C$  and  $L$  are known constant matrices of appropriate dimensions. Also, in the sequel of the work,  $P^+$  denotes the generalized inverse of matrix  $P$  satisfying  $PP^+P = P$ . The symbol  $P^\perp$  denotes a maximal row rank matrix such that  $P^\perp P = 0$ . When  $P$  is a full row rank matrix,  $P^\perp = 0$ .

**Remark 5** *This is the case of Standard LTI systems, where  $E = I$*

Now, let us consider a class of Linear Time-Invariant (LTI) systems of the form (3.59). The reconstruction of the function  $z(t)$  in finite-time can be obtained by using two functional generalized dynamical observers of the form:

$$\begin{aligned}
\dot{\zeta}_i(t) &= N_i\zeta_i(t) + J_iv_i(t) + F_iy(t) + H_iu(t) \\
\dot{v}_i(t) &= S_i\zeta_i(t) + G_iv_i(t) + M_iy(t) \\
\hat{z}_i(t) &= P_i\zeta_i(t) + Q_iy(t)
\end{aligned} \tag{3.60}$$

for  $i = 1, 2$ , where  $\zeta_i(t) \in \mathbb{R}^{q_0}$  are the states of the observers,  $v_i(t) \in \mathbb{R}^{q_1}$  are auxiliary vectors and  $\hat{z}_i(t) \in \mathbb{R}^q$  are the estimates of  $z(t)$ .  $N_i$ ,  $J_i$ ,  $F_i$ ,  $H_i$ ,  $S_i$ ,  $G_i$ ,  $M_i$ ,  $P_i$  and  $Q_i$  are constant matrices of appropriate dimensions to be determined. Both observers have the same structure but different parameter matrices. Before continuing, we make the following remark.

### Finite-Time Generalized Functional Observer Design

Consider parameter matrices  $T_i \in \mathbb{R}^{q_0 \times n}$  and define the transformed error vectors

$$\varepsilon_i(t) = \zeta_i(t) - T_ix(t) \tag{3.61}$$

The derivative of  $\varepsilon_i(t)$  is given by

$$\dot{\varepsilon}_i(t) = N_i\varepsilon_i(t) + (N_iT_i + F_iC - T_iA)x(t) + (H_i - T_iB)u(t) + J_iv_i(t) \tag{3.62}$$

$\dot{v}_i(t)$  in (3.60) can be rewritten as

$$\dot{v}_i(t) = S_i\varepsilon_i(t) + G_iv_i(t) + (M_iC + S_iT_i)x(t) \tag{3.63}$$

So, if the following conditions are satisfied

$$\mathbf{a.} \quad N_iT_i + F_iC - T_iA = 0$$

**b.**  $H_i = T_i B$

**c.**  $S_i T_i + M_i C = 0$ , for  $i=1,2$

then (3.62) and (3.63) can be written as

$$\dot{\varepsilon}_i(t) = N_i \varepsilon_i(t) + J_i v_i(t) \quad (3.64)$$

$$\dot{v}_i(t) = S_i \varepsilon_i(t) + G_i v_i(t) \quad (3.65)$$

By defining an augmented state vector  $\vartheta_i(t) = \begin{bmatrix} \varepsilon_i(t) \\ v_i(t) \end{bmatrix}$ , (3.64) and (3.65) can be rewritten as:

$$\dot{\vartheta}_i(t) = \underbrace{\begin{bmatrix} N_i & J_i \\ S_i & G_i \end{bmatrix}}_{\mathbb{A}_i} \vartheta_i(t) \quad (3.66)$$

Now, from (3.60) we obtain

$$\hat{z}_i(t) = P_i \varepsilon_i(t) + (P_i T_i + Q_i C) x(t) \quad (3.67)$$

considering that condition

**d.**  $P_i T_i + Q_i C = L$ , for  $i=1,2$

is satisfied, then

$$e_i(t) = P_i \varepsilon_i(t) \quad (3.68)$$

where  $e_i(t) = \hat{z}_i(t) - z(t)$ .

It can be seen that, if matrix  $\mathbb{A}_i$  from (3.66) is Hurwitz, then  $\lim_{t \rightarrow \infty} \vartheta_i(t) = 0$  and  $\lim_{t \rightarrow \infty} e_i(t) = 0$ , for  $i = 1, 2$ .

### Observer parameterization

From conditions **a.-d.**, it can be seen that the design problem of the two observers is reduced to find the matrices  $N_i, J_i, F_i, H_i, S_i, G_i, M_i, P_i, Q_i$  and  $T_i$  such that  $\mathbb{A}_i$  is Hurwitz.

Define matrix  $\Sigma = \begin{bmatrix} R \\ C \end{bmatrix}$ . In this case there always exists two matrices  $T_i$  and  $K_i$  such that

$$T_i + K_i C = R \quad (3.69)$$

which can be rewritten as

$$T_i = R - K_i C \quad (3.70)$$

From condition **a.** and (3.69), we have:

$$\begin{aligned} N_i (R - K_i C) + F_i C - (R - K_i C) A &= 0 \\ N_i R + (F_i - N_i K_i) C + K_i C A &= R A \end{aligned} \quad (3.71)$$

let  $K_{i1} = F_i - N_i K_i$ , then (3.71) can be written as

$$\begin{bmatrix} N_i & K_{i1} \end{bmatrix} \Sigma = RA - K_i CA \quad (3.72)$$

the general solution to (3.72) is

$$\begin{bmatrix} N_i & K_{i1} \end{bmatrix} = (RA - K_i CA)\Sigma^+ - Y_{i1}(I - \Sigma\Sigma^+) \quad (3.73)$$

where  $Y_{i1}$  is a matrix of appropriate dimensions with arbitrary elements. Now,  $N_i$  and  $K_{i1}$  can be obtained as it follows

$$N_i = N_{11} - K_i N_{22} - Y_{i1} N_3 \quad (3.74)$$

$$K_{i1} = \tilde{K}_1 - K_i \tilde{K}_2 - Y_{i1} \tilde{K}_3 \quad (3.75)$$

where  $N_{11} = RA\Sigma^+ \begin{bmatrix} I \\ 0 \end{bmatrix}$ ,  $N_2 = CA\Sigma^+ \begin{bmatrix} I \\ 0 \end{bmatrix}$ ,  $N_3 = (I - \Sigma\Sigma^+) \begin{bmatrix} I \\ 0 \end{bmatrix}$ ,  $\tilde{K}_1 = RA\Sigma^+ \begin{bmatrix} 0 \\ I \end{bmatrix}$ ,  $\tilde{K}_2 = CA\Sigma^+ \begin{bmatrix} 0 \\ I \end{bmatrix}$  and  $\tilde{K}_3 = (I - \Sigma\Sigma^+) \begin{bmatrix} 0 \\ I \end{bmatrix}$ .

If we substitute (3.70) into conditions **c.** and **d.** we obtain

$$\begin{bmatrix} S_i(R - K_i C) + M_i C \\ P_i(R - K_i C) + Q_i C \end{bmatrix} = \begin{bmatrix} 0 \\ L \end{bmatrix} \quad (3.76)$$

Let  $K_{i2} = (M_i - S_i K_i)$  and  $K_{i3} = (Q_i - P_i K_i)$ , then (3.76) can be written as

$$\begin{bmatrix} S_i & K_{i2} \\ P_i & K_{i3} \end{bmatrix} \Sigma = \begin{bmatrix} 0 \\ L \end{bmatrix} \quad (3.77)$$

the general solution to (3.130) is given by

$$\begin{bmatrix} S_i & K_{i2} \\ P_i & K_{i3} \end{bmatrix} = \begin{bmatrix} 0 \\ L \end{bmatrix} \Sigma^+ - Y_i(I - \Sigma\Sigma^+) \quad (3.78)$$

where  $Y_i = \begin{bmatrix} Y_{i2} \\ Y_{i3} \end{bmatrix}$  is a matrix of appropriate dimensions with arbitrary elements. In this case, the particular solutions for  $S_i$ ,  $K_{i2}$ ,  $P_i$  and  $Q_i$  are given by

$$S_i = -Y_{i2} N_3 \quad (3.79)$$

$$M_i = S_i K_i - Y_{i2} \tilde{K}_3 \quad (3.80)$$

$$P_i = P_{11} - Y_{i3} N_3 \quad (3.81)$$

$$Q_i = P_i K_i + \tilde{K}_{31} - Y_{i3} \tilde{K}_3 \quad (3.82)$$

where  $P_{11} = L\Sigma^+ \begin{bmatrix} I \\ 0 \end{bmatrix}$  and  $\tilde{K}_{31} = L\Sigma^+ \begin{bmatrix} 0 \\ I \end{bmatrix}$ .

We can see that the design is independent of the parameter matrices  $Y_{i3}$ , which can take an arbitrarily value. For simplicity we can take  $Y_{i3} = 0$ , in this case  $P_1 = P_2 = P$ .

Now, by using (3.74) and (3.79), the error dynamics (3.66) can be rewritten as

$$\dot{\vartheta}_i(t) = (\mathbb{A}_{i1} - \mathbb{Y}_i \mathbb{A}_2) \vartheta_i(t) \quad (3.83)$$

where  $\mathbb{A}_{i1} = \begin{bmatrix} N_1 - K_i N_2 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $\mathbb{Y}_i = \begin{bmatrix} Y_{i1} & J_i \\ Y_{i2} & G_i \end{bmatrix}$  and  $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I \end{bmatrix}$ . The problem is now reduced to find matrices  $\mathbb{Y}_i$  and  $K_i$  such that matrices  $\mathbb{A}_i$  are Hurwitz. This can be reached by using any pole placement method which can be expressed in a linear matrix inequality (LMI) approach.

Now, the necessary and sufficient condition for the existence of matrices  $\mathbb{Y}_i$  such that matrices  $\mathbb{A}_i$  are Hurwitz is that the pair  $(\mathbb{A}_2, \mathbb{A}_{i1})$  must be detectable.

The following Lemma gives the necessary and sufficient conditions for the detectability of the pair  $(\mathbb{A}_2, \mathbb{A}_{i1})$ .

### Generalized Functional Dynamical Observer design

From the precedent results, the design of a generalized finite time observer is reduced to determine the parameters  $\mathbb{Y}_i$  and  $K_i$  such that the eigenvalues of matrices  $\mathbb{A}_i$  are inside two disks regions  $\mathcal{Q}_1(\alpha_1, \tau_1)$  and  $\mathcal{Q}_2(\alpha_2, \tau_2)$  by using Lemma 3.

The following theorem gives the sufficient conditions in LMIs formulation to realize this objective.

**Theorem 2** *Under condition (3.34), let  $\mathcal{Q}_i(\alpha_i, \tau_i)$ ,  $i = 1, 2$  be two disks region, then there exist parameters matrices  $\mathbb{Y}_i$  and  $K_i$  such that the eigenvalues of matrices  $\mathbb{A}_i$  are inside these regions, if there exists two symmetric positive definite matrices  $X_i = \begin{bmatrix} X_{i1} & X_{i1} \\ X_{i1} & X_{i2} \end{bmatrix} \in \mathbb{R}^{(q_0+q_1) \times (q_0+q_1)}$  such that the following LMIs are satisfied:*

$$\begin{bmatrix} -X_i & \tilde{\mathbb{A}}_i \\ (*) & -N_3^{T\perp} X_{i1} N_3^{T\perp T} \end{bmatrix} < 0 \quad (3.84)$$

where  $\tilde{\mathbb{A}}_i = \frac{1}{\tau_i} \begin{bmatrix} (X_{i1} N_1 - W_i N_2 - \alpha_i X_{i1}) N_3^{T\perp T} \\ (X_{i1} N_1 - W_i N_2 - \alpha_i X_{i1}) N_3^{T\perp T} \end{bmatrix}$ ,  $W_i = X_{i1} K_i$  and matrices  $\mathbb{Y}_i$  can be parameterized as it follows:

$$\mathbb{Y}_i = -\tau_i (\mathcal{B}_{ir}^+ \mathcal{K}_i \mathcal{C}_l^+ + \mathcal{Z} - \mathcal{B}_{ir}^+ \mathcal{B}_{ir} \mathcal{Z}_i \mathcal{C}_l \mathcal{C}_l^+) \quad (3.85)$$

where

$$\mathcal{K}_i = -\mathcal{R}_i^{-1} \mathcal{B}_{il}^T \Theta_i \mathcal{C}_r^T (\mathcal{C}_r \Theta_i \mathcal{C}_r^T)^{-1} + \mathcal{S}_i^{1/2} \mathcal{L}_i (\mathcal{C}_r \Theta_i \mathcal{C}_r^T)^{-1/2} \quad (3.86)$$

$$\mathcal{S}_i = \mathcal{R}_i^{-1} - \mathcal{R}_i^{-1} \mathcal{B}_{il}^T [\Theta_i - \Theta_i \mathcal{C}_r^T (\mathcal{C}_r \Theta_i \mathcal{C}_r^T)^{-1} \mathcal{C}_r \Theta_i] \mathcal{B}_{il} \mathcal{R}_i^{-1} \quad (3.87)$$

and where  $\mathcal{L}_i$  is an arbitrary matrix such that  $\|\mathcal{L}_i\| < 1$  and  $\mathcal{R}_i$  is an arbitrary positive definite matrix such that

$$\Theta_i = (\mathcal{B}_{il} \mathcal{R}_i^{-1} \mathcal{B}_{il}^T - \mathcal{Q}_i)^{-1} > 0$$

for  $i = 1, 2$ .

**Proof 4** Under Assumption 1 and Condition (3.34), from Lemma 3 the eigenvalues of matrices  $\mathbb{A}_i$  are inside the two disks region  $\mathcal{Q}_i(\alpha_i, \tau_i)$ ,  $i = 1, 2$ , if the following LMIs are satisfied

$$\begin{bmatrix} -X_i & \frac{1}{\tau_i}(X_i\mathbb{A}_i - \alpha_i X_i) \\ (*) & -X_i \end{bmatrix} < 0 \quad (3.88)$$

where the symbol  $(*)$  denotes the transpose elements in the symmetric positions. Also (3.88) can be written as

$$\mathcal{B}_i \mathcal{X}_i \mathcal{C} + (\mathcal{B}_i \mathcal{X}_i \mathcal{C})^T + \mathcal{Q}_i < 0 \quad (3.89)$$

where  $\mathcal{B}_i = \begin{bmatrix} X_i \\ 0 \end{bmatrix}$ ,  $\mathcal{X}_i = -\frac{1}{\tau_i} \mathbb{Y}_i$ ,  $\mathcal{C} = [0 \quad \mathbb{A}_2]$ , and  $\mathcal{Q}_i = \begin{bmatrix} -X_i & \frac{1}{\tau_i}(X_i\mathbb{A}_{i1} - \alpha_i X_i) \\ (*) & -X_i \end{bmatrix}$ . From Lemma 3, the solvability conditions of (3.89) are

$$\mathcal{B}_i^\perp \mathcal{Q}_i \mathcal{B}_i^{\perp T} < 0 \quad (3.90)$$

$$\mathcal{C}^{T\perp} \mathcal{Q}_i \mathcal{C}^{T\perp T} < 0 \quad (3.91)$$

since  $\mathcal{B}_i^\perp = [0 \quad I]$ , inequality (3.90) becomes  $X_i > 0$  and considering  $\mathcal{C}^{T\perp} = \begin{bmatrix} I & 0 \\ 0 & \mathbb{A}_2^{T\perp} \end{bmatrix}$  we obtain

$$\begin{bmatrix} -X_i & \frac{1}{\tau_i}(X_i\mathbb{A}_{i1} - \alpha_i X_i)\mathbb{A}_2^{T\perp T} \\ (*) & -\mathbb{A}_2^{T\perp} X_i \mathbb{A}_2^{T\perp T} \end{bmatrix} < 0 \quad (3.92)$$

Now, it follows that

$$X_i\mathbb{A}_{i1} - \alpha_i X_i = \begin{bmatrix} X_{i1}N_1 - W_iN_2 - \alpha_i X_{i1} & -\alpha_i X_{i1} \\ X_{i1}N_1 - W_iN_2 - \alpha_i X_{i1} & -\alpha_i X_{i2} \end{bmatrix} \quad (3.93)$$

where  $W_i = X_{i1}K_i$ . Since,  $\mathbb{A}_2^{T\perp} = [N_3^{T\perp} \quad 0]$ , then  $(X_i\mathbb{A}_{i1} - \alpha_i X_i)\mathbb{A}_2^{T\perp T}$  can be written as

$$(X_i\mathbb{A}_{i1} - \alpha_i X_i)\mathbb{A}_2^{T\perp T} = \begin{bmatrix} (X_{i1}N_1 - W_iN_2 - \alpha_i X_{i1})N_3^{T\perp T} \\ (X_{i1}N_{i1} - W_iN_2 - \alpha_i X_{i1})N_3^{T\perp T} \end{bmatrix}$$

Let  $\tilde{\mathbb{A}}_i = \frac{1}{\tau_i} \begin{bmatrix} (X_{i1}N_1 - W_iN_2 - \alpha_i X_{i1})N_3^{T\perp T} \\ (X_{i1}N_{i1} - W_iN_2 - \alpha_i X_{i1})N_3^{T\perp T} \end{bmatrix}$ , then (3.92) can be written as

$$\begin{bmatrix} -X_i & \tilde{\mathbb{A}}_i \\ \tilde{\mathbb{A}}_i^T & -N_3^{T\perp} X_{i1} N_3^{T\perp T} \end{bmatrix} < 0 \quad (3.94)$$

By using Lemma 7 we obtain the parameterization of matrix  $\mathbb{Y}_i$  as in (3.85) which completes the proof.

**Remark 6** In order to achieve the pole placement of the observers, the pair  $(\mathbb{A}_2, \mathbb{A}_{i1})$  must be observable

### Finite-time Convergent Observer

Now that we have two Functional Generalized Dynamical Observers of the form (3.60), we can design the finite-time convergent observer. Let  $e(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}$  be the augmented estimation error consisting of the estimation error vectors from the FGDO's,  $\hat{z}(t) = \begin{bmatrix} \hat{z}_1(t) \\ \hat{z}_2(t) \end{bmatrix}$  and  $\vartheta(t) = \begin{bmatrix} \vartheta_1(t) \\ \vartheta_2(t) \end{bmatrix}$  then the augmented estimation error  $e(t) = \hat{z}(t) - \Psi z(t)$ , where  $\Psi = \begin{bmatrix} I & I \end{bmatrix}$ , can be written as

$$e(t) = \bar{P}\vartheta(t) \quad (3.95)$$

Where  $\bar{P} = \begin{bmatrix} \tilde{P} & 0 \\ 0 & \tilde{P} \end{bmatrix}$  and  $\tilde{P} = [P \ 0]$

By solving Eq. (3.66), we have

$$\vartheta(t) = e^{\mathbb{A}(t-t_0)}\vartheta(t_0) \quad (3.96)$$

where  $\mathbb{A} = \begin{bmatrix} \mathbb{A}_1 & 0 \\ 0 & \mathbb{A}_2 \end{bmatrix}$ , and

$$\begin{aligned} \vartheta(t-h) &= e^{\mathbb{A}(t-h-t_0)}\vartheta(t_0) \\ \implies \vartheta(t_0) &= e^{-\mathbb{A}(t-h-t_0)}\vartheta(t-h) \end{aligned} \quad (3.97)$$

By substituting (3.97) in (3.96) we have

$$\vartheta(t) = e^{\mathbb{A}h}\vartheta(t-h) \quad (3.98)$$

Now we pre-multiply (3.98) by  $\bar{P}$ , then, using (3.95), we have

$$e(t) = \hat{z}(t) - \Psi z(t) = \bar{P}e^{\mathbb{A}h}\vartheta(t-h) \quad (3.99)$$

from (3.95), we have

$$\bar{P}\vartheta(t-h) = e(t-h) \quad (3.100)$$

which can be solved by

$$\vartheta(t-h) = \bar{P}^+e(t-h) + (I - \bar{P}^+\bar{P})\rho \quad (3.101)$$

where  $\rho$  is an arbitrary vector with appropriate dimension. For simplicity, we take  $\rho = 0$ , now we substitute (3.101) in (3.99)

$$\hat{z}(t) - \Psi z(t) = \bar{P}e^{\mathbb{A}h}\bar{P}^+(\hat{z}(t-h) - \Psi z(t-h)) \quad (3.102)$$

which can be rewritten as

$$\Psi z(t) - \bar{Q}\Psi z(t-h) = \hat{z}(t) - \bar{Q}\hat{z}(t-h) \quad (3.103)$$

where  $\bar{Q} = \bar{P}e^{\mathbb{A}h}\bar{P}^+$  and

$$[\Psi \quad \bar{Q}\Psi] \begin{bmatrix} z(t) \\ -z(t-h) \end{bmatrix} = \hat{z}(t) - \bar{Q}\hat{z}(t-h) \quad (3.104)$$

Then,  $z(t)$  can be obtained by

$$\hat{z}(t) = [I \quad 0] [\Psi \quad \bar{Q}\Psi]^{-1} (\hat{z}(t) - \bar{Q}\hat{z}(t-h)) \quad (3.105)$$

By making

$$T_f = [I \quad 0] [\Psi \quad \bar{Q}\Psi]^{-1} \quad (3.106)$$

We can estimate the function by using

$$\hat{z}(t) = T_f (\hat{z}(t) - \bar{Q}\hat{z}(t-h)) \quad (3.107)$$

The existence of a finite-time convergence observer can be determined by matrix  $[\Psi \quad \bar{Q}\Psi]^{-1}$ , in this case, this matrix exists if

$$\begin{aligned} \det [\Psi \quad \bar{Q}\Psi] &\neq 0 \\ \det \begin{bmatrix} I & \bar{Q}_1 \\ I & \bar{Q}_2 \end{bmatrix} &\neq 0 \\ \det(\bar{Q}_1 - \bar{Q}_2) &\neq 0 \end{aligned} \quad (3.108)$$

where  $\bar{Q}_1 = \tilde{P}e^{\mathbb{A}_1 h}\tilde{P}^+$  and  $\bar{Q}_2 = \tilde{P}e^{\mathbb{A}_2 h}\tilde{P}^+$ .

**Remark 7** *The necessary condition for (3.108) to be satisfied is reduced to  $\lambda(\mathbb{A}_1) < \rho < \lambda(\mathbb{A}_2)$*

### Numerical example

To illustrate the finite-time convergent functional generalized observer approach, consider an LTI system of the form (3.59) with:

$$A = \begin{bmatrix} -1.0 & 0 & 0 & 1.0 & 3.0 \\ 2.0 & -5.0 & 0 & 1.0 & 2.0 \\ 0 & 0 & -4.0 & 1.0 & 3.0 \\ 1.0 & 0 & 0 & -2.0 & 4.0 \\ 0 & 0 & 0 & -2.0 & -4.0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ -1 & 0.5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1.0 & 0 & 0 & 1.0 & 0 \\ 0 & 1.0 & 1.0 & 0 & 1.0 \end{bmatrix},$$

$$\text{and } L = \begin{bmatrix} 1.0 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{We choose } R = \begin{bmatrix} 1.0 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 & 0 \\ 1.0 & 1.0 & 0 & 0 & 1.0 \\ 1.0 & 0 & 1.0 & 0 & 0 \end{bmatrix}$$

### Simulation 2 *Finite-time convergent state estimator for an LTI system*

The objective of this simulation is to estimate the hidden state of a dynamical system in a given convergence time.

**Settings.** We design two functional observers of the form (3.60) with their disks regions  $Q(\alpha_i, \tau_i)$  chosen as  $\alpha_1 = -7$ ,  $\tau_1 = 2$ ,  $\alpha_2 = -1$ ,  $\tau_2 = 0.5$ , and a finite-time convergent estimator of the form (3.107) with a convergence time of 0.5s and initial condition  $x(0) = [2 \ 1.5 \ -1 \ 0 \ 1]$ . The results are shown in Figure 3.4.

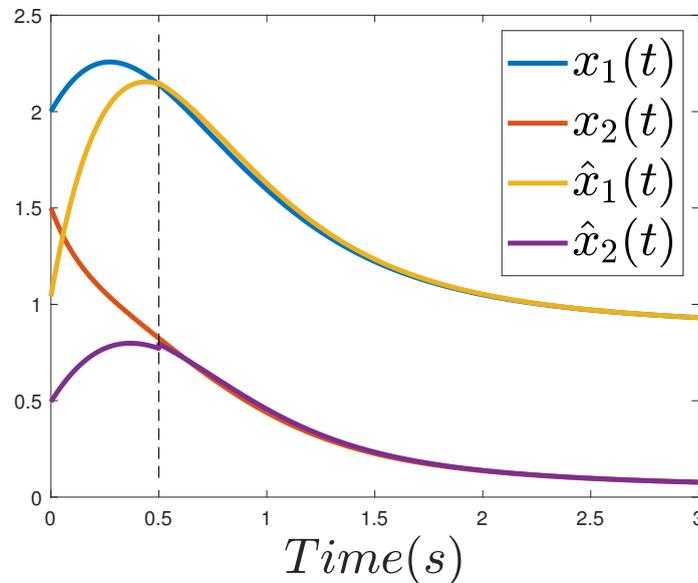


Figure 3.4: Estimated states

As we can see in Figure 3.4, the proposed functional observer is capable to estimate a partial state in the given convergence time.

## 3.2 Functional observer for descriptor systems with unknown input

This section presents a structure for a generalized functional observer prone to unknown inputs in the differential equations, for a class of descriptor linear systems of the form:

$$E\dot{x}(t) = Ax(t) + Bu(t) + Dd(t) \quad (3.109a)$$

$$y(t) = Cx(t) \quad (3.109b)$$

$$z(t) = Lx(t) \quad (3.109c)$$

where  $x(t) \in \mathbb{R}^n$  is the state of the system,  $u(t) \in \mathbb{R}^m$  is the known input,  $d(t) \in \mathbb{R}^l$  is the unknown input,  $z(t) \in \mathbb{R}^q$  is the function of the state aimed to be estimated, and  $y(t) \in \mathbb{R}^p$  is the measured output. This is an extension of the works made by [Osorio-Gordillo et al., 2016]. The necessary conditions for the existence and the determination of the observers gains are shown.

### 3.2.1 Problem statement

We consider the following generalized functional observer (GFO) of the form

$$\begin{aligned}\dot{\zeta}(t) &= N\zeta(t) + Jv(t) + F \begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} + Hu(t) + \Delta \hat{d}(t) \\ \dot{v}(t) &= S\zeta(t) + Gv(t) + M \begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} \\ \dot{\hat{d}}(t) &= \Phi\zeta(t) - Wy(t) \\ \hat{z}(t) &= P\zeta(t) + Q \begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix}\end{aligned}\tag{3.110}$$

where  $\zeta(t) \in \mathbb{R}^{q_0}$  is the state of the observer,  $v(t) \in \mathbb{R}^{q_1}$  is an auxiliary vector,  $\hat{d}(t)$  is the estimate of the unknown input, and  $\hat{z}(t) \in \mathbb{R}^q$  is the estimate of  $z(t)$ .  $N, J, F, H, \Delta, S, G, M, \Phi, W, P$  and  $Q$  are constant matrices of appropriate dimensions to be determined such that  $\lim_{t \rightarrow \infty} (\hat{z}(t) - z(t)) = 0$ .

The following Lemma gives the necessary conditions for the existence of the observer (3.110).

**Lemma 9** *There exists an observer having the form given in (3.110) for the system (3.109) if the matrix*

$$\begin{bmatrix} N & J & \Delta \\ S & G & 0 \\ \Phi & 0 & 0 \end{bmatrix}$$

*is Hurwitz and if there exists a matrix  $T$  such that the following conditions are satisfied:*

a.  $NTE + F \begin{bmatrix} E^\perp A \\ C \end{bmatrix} - TA = 0$

b.  $H = TB$

c.  $\Delta = TD$

d.  $STE + M \begin{bmatrix} E^\perp A \\ C \end{bmatrix} = 0$

e.  $\Phi TE - WC = 0$

$$f. PTE + Q \begin{bmatrix} E^\perp A \\ C \end{bmatrix} = L$$

**Proof 5** Consider a parameter matrix  $T \in \mathbb{R}^{q_0 \times n}$  and define the transformed error vector  $\varepsilon(t) = \zeta(t) - TE x(t)$ . Since  $E^\perp E = 0$  and  $E^\perp D = 0$  it can be deduced that  $E^\perp A x(t) = -E^\perp B u(t)$ , such that the derivative of  $\varepsilon(t)$  is given by

$$\dot{\varepsilon}(t) = N\zeta(t) + Jv(t) + \left( F \begin{bmatrix} E^\perp A \\ C \end{bmatrix} - TA \right) x(t) + (H - TB)u(t) + \Delta \hat{d}(t) - TDd(t) \quad (3.111)$$

By using the definition of  $\varepsilon(t)$ ,  $\dot{v}(t)$  in (3.60) can be rewritten as

$$\dot{v}(t) = S\zeta(t) + Gv(t) + M \left( F \begin{bmatrix} E^\perp A \\ C \end{bmatrix} - TA \right) x(t) \quad (3.112)$$

By considering the estimation error of the unknown input  $\tilde{d}(t) = \hat{d}(t) - d(t)$ , the estimation error of the unknown input can be written as

$$\dot{\tilde{d}}(t) = \Phi \zeta(t) - WCx(t) \quad (3.113)$$

Considering that  $\dot{d}(t) = 0$  and conditions a.-c. are satisfied, equations (3.111), (3.112) and (3.113) become

$$\dot{\varepsilon}(t) = N\varepsilon(t) + Jv(t) + \Delta \tilde{d}(t) \quad (3.114)$$

$$\dot{v}(t) = S\varepsilon(t) + Gv(t) \quad (3.115)$$

$$\dot{\tilde{d}}(t) = \Phi \varepsilon(t) \quad (3.116)$$

By defining an augmented state vector  $\sigma(t) = \begin{bmatrix} \varepsilon(t) \\ v(t) \\ \tilde{d}(t) \end{bmatrix}$ , equations (3.114), (3.115) can be rewritten as:

$$\dot{\sigma}(t) = \mathbb{A}\sigma(t) \quad (3.117)$$

where

$$\mathbb{A} = \begin{bmatrix} N & J & \Delta \\ S & G & 0 \\ \Phi & 0 & 0 \end{bmatrix}$$

Define the estimation error  $e(t) = z(t) - \hat{z}(t)$ . If condition c. is satisfied, then

$$e(t) = P\varepsilon(t)$$

It can be seen that if matrix  $\mathbb{A}$  is Hurwitz, then  $\lim_{t \rightarrow \infty} \varepsilon(t) = 0$  and  $\lim_{t \rightarrow \infty} e(t) = 0$ .

### 3.2.2 Observer parameterization

From Lemma 9, it can be deduced that the design of the observer is reduced to find the matrices  $N, J, F, H, \Delta, S, G, M, \Phi, W, P, Q$  and  $T$  such that conditions *a.-c.* are satisfied.

We define now matrix  $\Gamma = \begin{bmatrix} E \\ E^\perp A \\ C \end{bmatrix}$  and let  $R \in \mathbb{R}^{q_0 \times n}$  be a full row rank matrix such that  $\text{rank} \begin{bmatrix} R \\ \Gamma \end{bmatrix} = \text{rank}(\Gamma)$ . There always exists two matrices  $T$  and  $K$  such that

$$TE + K \begin{bmatrix} E^\perp A \\ C \end{bmatrix} = R \quad (3.118)$$

which can be rewritten as

$$[T \quad K] \Gamma = R \quad (3.119)$$

the general solution for (3.119) is

$$[T \quad K] = R\Gamma^+ - Z(I - \Gamma\Gamma^+) \quad (3.120)$$

which can be decomposed in

$$T = T_1 - ZT_2 \quad (3.121)$$

$$K = K_1 - ZK_2 \quad (3.122)$$

where  $Z$  is a constant matrix of appropriate dimension, and

$$T_1 = R\Gamma^+ \begin{bmatrix} I \\ 0 \end{bmatrix}$$

$$T_2 = (I - \Gamma\Gamma^+) \begin{bmatrix} I \\ 0 \end{bmatrix}$$

$$K_1 = R\Gamma^+ \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$K_2 = (I - \Gamma\Gamma^+) \begin{bmatrix} 0 \\ I \end{bmatrix}$$

Now we define matrix  $\Sigma = \begin{bmatrix} R \\ E^\perp A \\ C \end{bmatrix}$ . From condition *a.* from Lemma 9 and (3.118), we have

$$N \left( R - K \begin{bmatrix} E^\perp A \\ C \end{bmatrix} \right) + F \begin{bmatrix} E^\perp A \\ C \end{bmatrix} = TA \quad (3.123)$$

which can be written as

$$[N \quad \tilde{K}] \Sigma = TA \quad (3.124)$$

where  $\tilde{K} = F - NK$ . The necessary and sufficient condition for the existence of a solution to (3.124) is

$$\text{rank} \begin{bmatrix} \Sigma \\ TA \end{bmatrix} = \text{rank}(\Sigma)$$

the general solution to (3.124) is

$$[N \quad \tilde{K}] = TA\Sigma^+ - Y_1(I - \Sigma\Sigma^+) \quad (3.125)$$

if we replace (3.121) in (3.125), we obtain

$$N = N_1 - ZN_2 - Y_1N_3 \quad (3.126)$$

$$\tilde{K} = \tilde{K}_1 - Z\tilde{K}_2 - Y_1\tilde{K}_3 \quad (3.127)$$

where

$$N_1 = T_1A\Sigma^+ \begin{bmatrix} I \\ 0 \end{bmatrix}$$

$$N_2 = T_2A\Sigma \begin{bmatrix} I \\ 0 \end{bmatrix}$$

$$N_3 = (I - \Sigma\Sigma^+) \begin{bmatrix} I \\ 0 \end{bmatrix}$$

$$\tilde{K}_1 = T_1A\Sigma^+ \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$\tilde{K}_2 = T_2A\Sigma^+ \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$\tilde{K}_3 = (I - \Sigma\Sigma^+) \begin{bmatrix} 0 \\ I \end{bmatrix}$$

since  $F = \tilde{K} + NK$ , we have

$$\begin{aligned} F &= \tilde{K}_1 + N_1K - Z(\tilde{K}_2 - N_2K) - Y_1(\tilde{K}_3 - N_3K) \\ F &= F_1 - ZF_2 - Y_1F_3 \end{aligned} \quad (3.128)$$

where

$$F_1 = T_1A\Sigma^+ \begin{bmatrix} K \\ I \end{bmatrix}$$

$$F_2 = T_2A\Sigma^+ \begin{bmatrix} K \\ I \end{bmatrix}$$

$$F_3 = (I - \Sigma\Sigma^+) \begin{bmatrix} K \\ I \end{bmatrix}$$

from (3.118) we have

$$\begin{bmatrix} TE \\ E^\perp A \\ C \end{bmatrix} = \begin{bmatrix} I & -K \\ 0 & I \end{bmatrix} \Sigma \quad (3.129)$$

Conditions *c.* and *d.* of Lemma 9 can be written as

$$\begin{bmatrix} S & M \\ P & Q \end{bmatrix} \begin{bmatrix} TE \\ E^\perp A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ L \end{bmatrix} \quad (3.130)$$

if we substitute equation (3.129) into (3.130), we obtain

$$\begin{bmatrix} S & M \\ P & Q \end{bmatrix} \begin{bmatrix} I & -K \\ 0 & I \end{bmatrix} \Sigma = \begin{bmatrix} 0 \\ L \end{bmatrix} \quad (3.131)$$

The necessary and sufficient condition for the existence of a solution to (3.131) is

$$\text{rank} \begin{bmatrix} \Sigma \\ L \end{bmatrix} = \text{rank}(\Sigma)$$

and since

$$\begin{bmatrix} I & -K \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & K \\ 0 & I \end{bmatrix}$$

the general solution is given by

$$\begin{bmatrix} S & M \\ P & Q \end{bmatrix} = \left( \begin{bmatrix} 0 \\ L \end{bmatrix} \Sigma^+ - Y(I - \Sigma \Sigma^+) \right) \begin{bmatrix} I & K \\ 0 & I \end{bmatrix} \quad (3.132)$$

where  $Y = \begin{bmatrix} Y_2 \\ Y_3 \end{bmatrix}$  is an arbitrary matrix of appropriate dimensions. In this case the particular solutions for  $S$ ,  $M$ ,  $P$  and  $Q$  are given by

$$S = -Y_2 N_3 \quad (3.133)$$

$$M = -Y_2 F_3 \quad (3.134)$$

$$P = P_1 - Y_3 N_3 \quad (3.135)$$

$$Q = Q_1 - Y_3 F_3 \quad (3.136)$$

where

$$P_1 = L \Sigma^+ \begin{bmatrix} I \\ 0 \end{bmatrix}$$

$$Q_1 = L \Sigma^+ \begin{bmatrix} K \\ I \end{bmatrix}.$$

Condition *e.* of Lemma 9 can be written as

$$\begin{aligned}\Phi \left( R - K \begin{bmatrix} E^\perp A \\ C \end{bmatrix} \right) - WC &= 0 \\ \Phi R - \bar{K} \begin{bmatrix} E^\perp A \\ C \end{bmatrix} &= WC\end{aligned}$$

$$[\Phi \quad -\bar{K}] \Sigma = WC \quad (3.137)$$

The general solution for (3.137) is

$$[\Phi \quad -\bar{K}] = WC\Sigma^+ - Y_4(I - \Sigma\Sigma^+) \quad (3.138)$$

(3.138) can be rewritten as

$$\Phi = \Phi_1 - Y_4 N_3 \quad (3.139)$$

$$\bar{K} = \bar{K}_1 + Y_4 K_2 \quad (3.140)$$

where  $W$  is an arbitrary matrix of appropriate dimensions,  $\Phi_1 = WC\Sigma^+ \begin{bmatrix} I \\ 0 \end{bmatrix}$  and  $\bar{K}_1 = WC\Sigma^+ \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

Now, by using (3.126), (3.133) and (3.139), the error dynamics (3.117) can be rewritten as

$$\dot{\sigma}(t) = (\mathbb{A}_1 - \mathbb{Y}\mathbb{A}_2)\sigma(t) \quad (3.141)$$

where

$$\mathbb{A}_1 = \begin{bmatrix} N_1 - ZN_2 & 0 & \Delta \\ 0 & 0 & 0 \\ \Phi & 0 & 0 \end{bmatrix}, \quad \mathbb{Y} = \begin{bmatrix} Y_1 & J & 0 \\ Y_2 & G & 0 \\ Y_4 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbb{A}_2 = \begin{bmatrix} N_3 & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The problem has been now reduced to find matrices  $\mathbb{Y}$  and  $Z$  such that the real parts of the eigenvalues of the matrix  $\mathbb{A}$  are negative. This can be reached by using the linear matrix inequality (LMI) approach.

### 3.2.3 Observer design

The following theorem gives the LMI conditions that will allow to determine all the observer matrices.

**Theorem 3.2.1** *Under Assumption 1 there exist two matrices  $\mathbb{Y}$  and  $Z$  such that (3.83) is asymptotically stable if there exists a matrix*

$$X = \begin{bmatrix} X_1 & X_2 & X_3 \\ X_2^T & X_4 & X_5 \\ X_3^T & X_5^T & X_6 \end{bmatrix} > 0$$

such that the following LMI is satisfied:

$$N_3^{T\perp} (N_1^T X_1 + X_1 N_1 - N_2^T W^T - W N_2 + X_3 \Phi_1 + \Phi_1 X_3^T) N_3^{T\perp T} < 0. \quad (3.142)$$

Where  $Z = X_1^{-1}W$ ,  $X_3$  is an arbitrary matrix of appropriate dimensions, and matrix  $\mathbb{Y}$  is determined as follows

$$\mathbb{Y} = -X^{-1}(-\gamma \mathcal{B}^T + \sqrt{\gamma} \mathcal{L} \Omega^{1/2})^T \quad (3.143)$$

where

$$\Omega = \gamma \mathcal{B} \mathcal{B}^T - \mathcal{Q} > 0 \quad (3.144)$$

with

$$\mathcal{Q} = \begin{bmatrix} X_1(N_1 - ZN_2) + (N_1 - ZN_2)^T X_1 + X_3 \Phi_1 + \Phi_1^T X_3^T & (N_1 - ZN_2)^T X_2 + \Phi_1^T X_5^T & 0 \\ X_2^T (N_1 - ZN_2) + X_5 \Phi_1 & 0 & 0 \\ X_3(N_1 - ZN_2) + X_6 \Phi + \Delta^T X_1 & 0 & 0 \\ X_1 \Delta + (N_1 - ZN_2)^T X_3 + \Phi^T X_6 \\ X_2^T \Delta \\ X_3 \Delta \end{bmatrix}$$

$$\mathcal{B} = \begin{bmatrix} N_3^T & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and matrix  $\mathcal{L}$  is any matrix such that  $\|\mathcal{L}\| < 1$  and  $\gamma > 0$  is any scalar such that  $\Omega > 0$ .

**Proof 6** We define the following Lyapunov function candidate

$$V(\sigma(t)) = \sigma(t)^T X \sigma(t) \quad (3.145)$$

its derivative is given by

$$\dot{V}(\sigma(t)) = \sigma(t)^T [(\mathbb{A}_1 - \mathbb{Y} \mathbb{A}_2)^T X + X(\mathbb{A}_1 - \mathbb{Y} \mathbb{A}_2)] \sigma(t) \quad (3.146)$$

The asymptotic stability of (3.141) is guaranteed only if  $\dot{V}(\sigma(t)) < 0$ , this leads to the following LMI

$$\mathbb{A}_1^T X - \mathbb{A}_2^T \mathbb{Y}^T X + X \mathbb{A}_1 - X \mathbb{Y} \mathbb{A}_2 < 0 \quad (3.147)$$

which can be rewritten as

$$\mathcal{Q} + \mathcal{B} \mathcal{X} + (\mathcal{B} \mathcal{X})^T < 0 \quad (3.148)$$

where  $\mathcal{X} = -\mathbb{Y}^T X$ ,  $\mathcal{Q} = X \mathbb{A}_1 + \mathbb{A}_1^T X$  and  $\mathcal{B} = \mathbb{A}_2^T$ . According to the Elimination Lemma, there exists a matrix  $\mathcal{X}$  satisfying (3.148) if and only if the following condition holds

$$\mathcal{B}^\perp \mathcal{Q} \mathcal{B}^{\perp T} < 0 \quad (3.149)$$

with  $\mathcal{B}^\perp = [N_3^{T\perp} \ 0 \ 0]$ . By using the definitions of  $\mathcal{Q}$  and  $W$ , we obtain (3.142). Matrix  $\mathbb{Y}$  is obtained from (3.143) and (3.144).

### 3.3 Functional observer for T-S descriptor systems

Consider a class of descriptor Takagi-Sugeno systems of the form:

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=1}^k w_i (A_i x(t) + B_i u(t)) \\ y(t) &= Cx(t) \end{aligned} \quad (3.150)$$

where  $x(t) \in \mathbb{R}^n$  is the state of the system,  $u(t) \in \mathbb{R}^m$  is the input vector and  $y(t) \in \mathbb{R}^p$  is the measured output of the system.

**Assumption 2** *Descriptor system (3.150) is impulse observable, i.e.*

$$\text{rank} \begin{bmatrix} E \\ E^\perp A_i \\ C \end{bmatrix} = n \quad (3.151)$$

this is equivalent to  $\text{rank} \begin{bmatrix} E & A_i \\ 0 & C \\ 0 & E \end{bmatrix} = \text{rank}(E) + n$

The following Lemma is presented in [Skelton et al., 1997] and will be used later.

**Lemma 10** *Let matrices  $\mathcal{B}$  and  $\mathcal{Q}$  be given. The following statements are equivalent:*

1. *There exists a matrix  $\mathcal{X}$  satisfying  $\mathcal{B}\mathcal{X} + (\mathcal{B}\mathcal{X})^T + \mathcal{Q} < 0$*
2. *The following condition holds*

$$\mathcal{B}^\perp \mathcal{Q} \mathcal{B}^{\perp T} < 0$$

*Suppose the above statements hold and assume that  $\mathcal{B}^\perp \mathcal{B} > 0$ . Then matrix  $\mathcal{X}$  in statement 1 is given by*

$$\mathcal{X} = -\gamma \mathcal{B}^T + \sqrt{\gamma} \mathcal{L} \Gamma^{1/2}$$

*where  $\mathcal{L}$  is any matrix such that  $\|\mathcal{L}\| < 1$  and  $\gamma > 0$  is any scalar such that*

$$\Gamma = \gamma \mathcal{B} \mathcal{B}^T - \mathcal{Q} > 0$$

#### 3.3.1 Problem statement

Consider the following generalized functional observer (GFO) of the form

$$\begin{aligned} \dot{\zeta}(t) &= \sum_{i=1}^k w_i (N_i \zeta(t) + J_i v(t) + F_i y(t) + H_i u(t)) \\ \dot{v}(t) &= \sum_{i=1}^k w_i (S_i \zeta(t) + G_i v(t) + M_i y(t)) \\ \hat{z}(t) &= \sum_{i=1}^k w_i (P_i \zeta(t) + Q_i y(t)) \end{aligned} \quad (3.152)$$

where  $\zeta(t) \in \mathbb{R}^{q_0}$  is the state of the observer,  $v(t) \in \mathbb{R}^{q_1}$  is an auxiliary vector and  $\hat{z}(t) \in \mathbb{R}^q$  is the estimate of  $z(t)$ .  $N_i, J_i, F_i, H_i, S_i, G_i, M_i, P_i$  and  $Q_i$  are constant matrices of appropriate dimensions to be determined such that  $\lim_{t \rightarrow \infty} (\hat{z}(t) - z(t)) = 0$ .

The following Lemma gives the sufficient conditions for the existence of the observer (3.152).

**Lemma 11** *There exists an observer having the form given in (3.152) for the system (3.150) if the matrix  $\begin{bmatrix} N_i & J_i \\ S_i & G_i \end{bmatrix}$  is Hurwitz and if there exists a matrix  $T$  such that the following conditions are satisfied:*

$$a. N_i T E + F_i C - T A_i = 0$$

$$b. H_i = T B_i$$

$$c. S_i T E + M_i C = 0$$

$$d. P_i T E + Q_i C = L$$

**Proof 7** *Consider a parameter matrix  $T \in \mathbb{R}^{q_0 \times n}$  and define the transformed error vector  $\varepsilon(t) = \zeta(t) - T E x(t)$ . Since  $E^\perp E = 0$  it can be deduced that  $E^\perp A_i x(t) = -E^\perp B_i u(t)$ , such that the derivative of  $\varepsilon(t)$  is given by*

$$\begin{aligned} \dot{\varepsilon}(t) &= N_i \zeta(t) + (F_i C - T A_i) x(t) \\ &\quad + (H_i - T B_i) u(t) + J_i v(t) \end{aligned} \quad (3.153)$$

By using the definition of  $\varepsilon(t)$ ,  $\dot{v}(t)$  in (3.152) can be rewritten as

$$\dot{v}(t) = S_i \zeta(t) + M_i (F_i C - T A_i) x(t) + G_i v(t) \quad (3.154)$$

Considering that conditions a.-d. are satisfied, equations (3.153) and (3.154) become

$$\dot{\varepsilon}(t) = N_i \varepsilon(t) + J_i v(t) \quad (3.155)$$

$$\dot{v}(t) = S_i \varepsilon(t) + G_i v(t) \quad (3.156)$$

By defining an augmented state vector  $\sigma(t) = \begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix}$ , equations (3.155), (3.156) can be rewritten as:

$$\dot{\sigma}(t) = \mathbb{A}_i \sigma(t) \quad (3.157)$$

where  $\mathbb{A}_i = \begin{bmatrix} N_i & J_i \\ S_i & G_i \end{bmatrix}$ .

Define the estimation error  $e(t) = z(t) - \hat{z}(t)$ . If condition c. is satisfied, then  $e(t) = P_i \varepsilon(t)$ . It can be seen that if matrix  $\mathbb{A}_i$  is Hurwitz, then  $\lim_{t \rightarrow \infty} \varepsilon(t) = 0$  and  $\lim_{t \rightarrow \infty} e(t) = 0$ .

### 3.3.2 Observer parameterization

From Lemma 11, it can be deduced that the design of the observer is reduced to find the matrices  $N_i, J_i, F_i, H_i, S_i, G_i, M_i, P_i, Q_i$  and  $T$  such that conditions *a.-d.* are satisfied. We define now matrix  $\Gamma = \begin{bmatrix} E \\ C \end{bmatrix}$  and let  $R \in \mathbb{R}^{q_0 \times n}$  be a full row rank matrix such that  $\text{rank} \begin{bmatrix} R \\ \Gamma \end{bmatrix} = \text{rank}(\Gamma)$ . In this case there always exists two matrices  $T$  and  $K$  such that,  $TE + KC = R$ , which can be rewritten as

$$[T \quad K] \Gamma = R \quad (3.158)$$

the general solution for (3.158) is

$$[T \quad K] = R\Gamma^+ - Z(I - \Gamma\Gamma^+) \quad (3.159)$$

which can be decomposed in

$$T = T_1 - Z_1 T_2 \quad (3.160)$$

$$K = K_1 - Z_1 K_2 \quad (3.161)$$

where  $Z$  is a constant matrix of appropriate dimension, and  $T_1 = R\Gamma^+ \begin{bmatrix} I \\ 0 \end{bmatrix}$ ,

$T_2 = (I - \Gamma\Gamma^+) \begin{bmatrix} I \\ 0 \end{bmatrix}$ ,  $K_1 = R\Gamma^+ \begin{bmatrix} 0 \\ I \end{bmatrix}$ , and  $K_2 = (I - \Gamma\Gamma^+) \begin{bmatrix} 0 \\ I \end{bmatrix}$ .

Now we define matrix  $\Sigma = \begin{bmatrix} R \\ C \end{bmatrix}$ . From condition *a.* from Lemma 11 and (3.158), we have

$$N_i(R - KC) + F_i C = T A_i \quad (3.162)$$

which can be written as

$$[N_i \quad \tilde{K}_i] \Sigma = T A_i \quad (3.163)$$

where  $\tilde{K} = F_i - N_i K$ . The necessary and sufficient condition for the existence of a solution to (3.163) is

$$\text{rank} \begin{bmatrix} \Sigma \\ T A_i \end{bmatrix} = \text{rank}(\Sigma)$$

the general solution to (3.163) is

$$[N_i \quad \tilde{K}_i] = T A_i \Sigma^+ - Y_{i1}(I - \Sigma\Sigma^+) \quad (3.164)$$

if we replace (3.160) in (3.164), we obtain

$$N_i = N_{i1} - Z N_{i2} - Y_{i1} N_3 \quad (3.165)$$

$$\tilde{K}_i = \tilde{K}_{i1} - Z \tilde{K}_{i2} - Y_{i1} \tilde{K}_{i3} \quad (3.166)$$

where  $N_{i1} = T_1 A_i \Sigma^+ \begin{bmatrix} I \\ 0 \end{bmatrix}$ ,  $N_{i2} = T_2 A_i \Sigma \begin{bmatrix} I \\ 0 \end{bmatrix}$ ,  $N_3 = (I - \Sigma\Sigma^+) \begin{bmatrix} I \\ 0 \end{bmatrix}$ ,  $\tilde{K}_{i1} = T_1 A_i \Sigma^+ \begin{bmatrix} 0 \\ I \end{bmatrix}$ ,

$$\tilde{K}_{i2} = T_2 A_i \Sigma^+ \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \tilde{K}_3 = (I - \Sigma \Sigma^+) \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Since  $F_i = \tilde{K}_i + N_i K$ , we have

$$\begin{aligned} F_i &= \tilde{K}_{i1} + N_{i1} K - Z(\tilde{K}_{i2} - N_{i2} K) - Y_{i1}(\tilde{K}_3 - N_3 K) \\ F_i &= F_{i1} - Z F_{i2} - Y_{i1} F_3 \end{aligned} \quad (3.167)$$

$$\text{where } F_{i1} = T_1 A_i \Sigma^+ \begin{bmatrix} K \\ I \end{bmatrix}, \quad F_{i2} = T_2 A_i \Sigma^+ \begin{bmatrix} K \\ I \end{bmatrix}, \quad F_{i3} = (I - \Sigma \Sigma^+) \begin{bmatrix} K \\ I \end{bmatrix}.$$

From (3.158) we have

$$\begin{bmatrix} TE \\ C \end{bmatrix} = \begin{bmatrix} I & -K \\ 0 & I \end{bmatrix} \Sigma \quad (3.168)$$

Conditions *c.* and *d.* of Lemma 11 can be written as

$$\begin{bmatrix} S_i & M_i \\ P_i & Q_i \end{bmatrix} \begin{bmatrix} TE \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ L \end{bmatrix} \quad (3.169)$$

replacing equation (3.168) into (3.169), we obtain

$$\begin{bmatrix} S_i & M_i \\ P_i & Q_i \end{bmatrix} \begin{bmatrix} I & -K \\ 0 & I \end{bmatrix} \Sigma = \begin{bmatrix} 0 \\ L \end{bmatrix} \quad (3.170)$$

since  $\begin{bmatrix} I & -K \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & K \\ 0 & I \end{bmatrix}$  the general solution is given by

$$\begin{bmatrix} S_i & M_i \\ P_i & Q_i \end{bmatrix} = \left( \begin{bmatrix} 0 \\ L \end{bmatrix} \Sigma^+ - Y_i (I - \Sigma \Sigma^+) \right) \begin{bmatrix} I & K \\ 0 & I \end{bmatrix} \quad (3.171)$$

where  $Y_i = \begin{bmatrix} Y_{i2} \\ Y_{i3} \end{bmatrix}$  is an arbitrary matrix of appropriate dimensions. In this case the particular solutions for  $S_i$ ,  $M_i$ ,  $P_i$  and  $Q_i$  are given by

$$S_i = -Y_{i2} N_3 \quad (3.172)$$

$$M_i = -Y_{i2} F_3 \quad (3.173)$$

$$P_i = P_{i1} - Y_{i3} N_3 \quad (3.174)$$

$$Q_i = Q_{i1} - Y_{i3} F_{i3} \quad (3.175)$$

$$\text{where } P_{i1} = L \Sigma^+ \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad Q_{i1} = L \Sigma^+ \begin{bmatrix} K \\ I \end{bmatrix}.$$

Now, by using (3.165) and (3.172), the error dynamics (3.157) it can be rewritten as

$$\dot{\sigma}(t) = (\mathbb{A}_{i1} - \mathbb{Y}_i \mathbb{A}_2) \sigma(t) \quad (3.176)$$

$$\text{where } \mathbb{A}_{i1} = \begin{bmatrix} N_{i1} - Z N_{i2} & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbb{Y} = \begin{bmatrix} Y_{i1} & J_i \\ Y_{i2} & G_i \end{bmatrix} \text{ and } \mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I \end{bmatrix}.$$

The problem has been now reduced to find matrices  $\mathbb{Y}_i$  and  $Z$  such that matrix  $\mathbb{A}_i = \mathbb{A}_{i1} - \mathbb{Y}_i \mathbb{A}_2$  is Hurwitz. This can be reached by using the linear matrix inequality (LMI) approach.

### 3.3.3 Functional observer design

**Theorem 3.3.1** *Under Assumption 1 there exist two matrices  $\mathbb{Y}_i$  and  $Z$  such that (3.176) is asymptotically stable if there exists a matrix*

$$X = \begin{bmatrix} X_1 & X_1 \\ X_1 & X_2 \end{bmatrix} > 0$$

such that the following LMI is satisfied:

$$N_3^{T\perp} (N_{i1}^T X_1 + X_1 N_{i1} - N_{i2}^T W^T - W N_{i2}) N_3^{T\perp T} < 0 \quad (3.177)$$

where  $Z = X_1^{-1}W$  and matrix  $\mathbb{Y}_i$  can be determined as follows

$$\mathbb{Y}_i = -X^{-1}(-\gamma \mathcal{B}^T + \sqrt{\gamma} \mathcal{L} \Omega_i^{1/2})^T \quad (3.178)$$

where

$$\Omega_i = \gamma \mathcal{B} \mathcal{B}^T - \mathcal{Q}_i > 0 \quad (3.179)$$

with

$$\mathcal{Q}_i = \begin{bmatrix} X_1(N_{i1} - ZN_{i2}) + (N_{i1} - ZN_{i2})^T X_1 & (N_{i1} - ZN_{i2}^T)X_1 \\ X_1(N_{i1} - ZN_{i2}) & 0 \end{bmatrix}$$

$$\mathcal{B} = \begin{bmatrix} N_3^T & 0 \\ 0 & -I \end{bmatrix}$$

and matrix  $\mathcal{L}$  is any matrix such that  $\|\mathcal{L}\| < 1$  and  $\gamma > 0$  is any scalar such that  $\Omega_i > 0$ .

**Proof 8** Consider the following Lyapunov function candidate

$$V(\sigma(t)) = \sigma(t)^T X_i \sigma(t) \quad (3.180)$$

which derivative is given by

$$\dot{V}(\sigma(t)) = \sigma(t)^T [(\mathbb{A}_{i1} - \mathbb{Y}_i \mathbb{A}_2)^T X_i + X(\mathbb{A}_{i1} - \mathbb{Y}_i \mathbb{A}_2)] \sigma(t) \quad (3.181)$$

The asymptotic stability of (3.176) is guaranteed only if  $\dot{V}(\sigma(t)) < 0$ , this leads to the following LMI

$$\mathbb{A}_{i1}^T X - \mathbb{A}_2^T \mathbb{Y}_i^T X + X \mathbb{A}_{i1} - X \mathbb{Y}_i \mathbb{A}_2 < 0 \quad (3.182)$$

which can be rewritten as

$$\mathcal{Q}_i + \mathcal{B} \mathcal{X}_i + (\mathcal{B} \mathcal{X}_i)^T < 0 \quad (3.183)$$

where  $\mathcal{X}_i = -\mathbb{Y}_i^T X$ ,  $\mathcal{Q}_i = X \mathbb{A}_{i1} + \mathbb{A}_{i1}^T X$  and  $\mathcal{B} = \mathbb{A}_2^T$ . According to Lemma 10, there exists a matrix  $\mathcal{X}_i$  satisfying (3.183) if and only if the following condition holds

$$\mathcal{B}^\perp \mathcal{Q}_i \mathcal{B}^{\perp T} < 0 \quad (3.184)$$

with  $\mathcal{B}^\perp = [N_3^{T\perp} \ 0]$ . By using the definitions of  $\mathcal{Q}_i$  and  $W$ , we obtain (3.177). Matrix  $\mathbb{Y}_i$  is obtained from (3.178).

The following algorithm summarize the observer design to obtain the corresponding matrices.

*Algorithm 1:*

1. Choose a matrix  $R$  such that  $\text{rank} \begin{bmatrix} R \\ \Gamma \end{bmatrix} = \text{rank}(\Gamma)$ .
2. Compute matrices  $N_{i1}$ ,  $N_{i2}$ ,  $N_3$ ,  $T_1$ ,  $T_2$ ,  $K_1$ ,  $K_2$ ,  $P_{i1}$  and  $Q_{i1}$ .
3. Solve the LMI (3.177) to find  $X$  and  $Z$ .
4. Choose a matrix  $\mathcal{L}$  such that  $\|\mathcal{L}\| < 1$ , and a scalar  $\gamma > 0$  such that  $\Omega_i > 0$ , then determinate matrix  $\mathbb{Y}_i$  as in (3.178).
5. Compute all the matrices gains of the observer (3.152) by using (3.165) to determinate  $N_i$ , (3.178) to determinate  $J_i$  and  $G_i$ , (3.172)-(3.175) to find  $S_i$ ,  $M_i$ ,  $P_i$  and  $Q_i$  taking matrix  $Y_{i3} = 0$ .  $F_i$  is given by (3.167) and matrix  $H_i$  could be determined with condition  $b$ .

### 3.3.4 Particular case: Proportional Functional observer

In order to obtain a Proportional Functional Observer (PFO) from the GFO, it corresponds to the parameter matrices  $S = 0$ ,  $J = 0$ ,  $M = 0$ ,  $G = 0$ ,  $F_i = \begin{bmatrix} 0 & F_{ai} \end{bmatrix}$  and  $Q_i = \begin{bmatrix} 0 & Q_{ai} \end{bmatrix}$ , which generates the following observer [Trinh and Fernando, 2011]:

$$\begin{aligned} \dot{\zeta}(t) &= \sum_{i=1}^k w_i (N_i \zeta(t) + F_{ai} y(t) + H_i u(t)) \\ \hat{z}(t) &= \sum_{i=1}^k w_i (P_i \zeta(t) + Q_{ai} y(t)) \end{aligned} \quad (3.185)$$

and the error dynamics (3.157) becomes

$$\dot{\varepsilon}(t) = \left( \tilde{\mathbb{A}}_{i1} - \tilde{\mathbb{Y}}_i \tilde{\mathbb{A}}_2 \right) \varepsilon(t) \quad (3.186)$$

where  $\tilde{\mathbb{A}}_{i1} = N_{i1} - ZN_{i2}$ ,  $\tilde{\mathbb{Y}}_i = Y_{i1}$  and  $\tilde{\mathbb{A}}_2 = N_3$ . Matrices  $\mathcal{Q}_i$ ,  $\mathcal{B}$  and  $\mathcal{X}_i$  of Theorem 1 become:

$$\begin{aligned} \mathcal{Q}_i &= X(N_{i1} - ZN_{i2}) + (N_{i1} - ZN_{i2})^T X, \\ \mathcal{B} &= N_3^T, \\ \mathcal{X}_i &= -Y_{i1}^T X \end{aligned}$$

Matrices  $\Gamma$  and  $\Sigma$  are defined as  $\Gamma = \begin{bmatrix} E \\ C \end{bmatrix}$  and  $\Sigma = \begin{bmatrix} R \\ C \end{bmatrix}$ . With these matrices, the observer matrices can be obtained by the following the Algorithm 1.

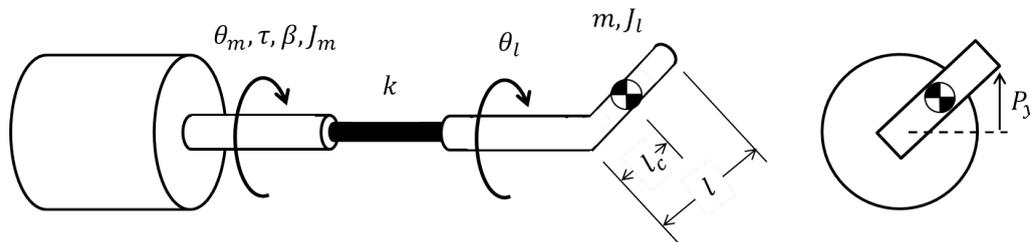


Figure 3.5: Single link robot arm

### 3.3.5 Mathematical model

The mathematical model chosen to test the performance of the generalized observer is a single link robot arm (see Fig. 3.5).

The model was selected from the one presented in [Chakrabarty et al., 2017] with the following nonlinear model:

$$\begin{aligned} J_m \ddot{\theta}_m(t) &= -\beta \dot{\theta}_m(t) - k(\theta_m(t) - \theta_l(t)) + K_\tau u(t) \\ J_l \ddot{\theta}_l(t) &= k(\theta_m(t) - \theta_l(t)) + l_c m g \sin \theta_l(t) \end{aligned} \tag{3.187}$$

The parameters of the model are given in Table 3.1

Table 3.1: Parameters of model

$J_m$	Moment of inertia of the motor shaft	0.01 Kg * m <sup>2</sup>
$J_l$	Moment of inertia of the link	0.08 Kg * m <sup>2</sup>
$m$	Mass of the link	0.15 m
$l_c$	Distance of the center of mass	0.25 m
$k$	Spring rigidity coefficient	0.3 Nm/rad
$\beta$	Viscous friction coefficient	0.05 Nms/rad
$K_\tau$	Constant of the motor	0.2 Nm/V
$l$	Lenght of the link	0.18 m

Taking as the states  $x_1(t) = \theta_m(t)$ ,  $x_2(t) = \dot{\theta}_m(t)$ ,  $x_3(t) = \theta_l(t)$ ,  $x_4(t) = \dot{\theta}_l(t)$  and  $x_5(t) = P_y(t)$ , we can represent (2.6) as it follows:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\frac{\beta}{J_m} x_2(t) - \frac{k}{J_m} x_1(t) + \frac{k}{J_m} x_3(t) + \frac{K_\tau}{J_m} u(t) \\ \dot{x}_3(t) &= x_4(t) \\ \dot{x}_4(t) &= \frac{k}{J_l} x_1(t) - \frac{k}{J_l} x_3(t) + \frac{l_c m g}{J_l} \sin x_3(t) \\ 0 &= l \sin x_3(t) - x_5(t) \end{aligned} \tag{3.188}$$

### Takagi-Sugeno modeling

We chose as the scheduling variable  $\rho(t) = \frac{\sin x_3(t)}{x_3(t)}$ , so the system (3.188) can be represented:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{k}{J_m} & -\frac{\beta}{J_m} & \frac{k}{J_m} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{k}{J_l} & 0 & \frac{1}{J_m} (k - l_c m g \rho(t)) & 0 & 0 \\ 0 & 0 & l \rho(t) & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{K_\tau}{J_m} \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t) \quad (3.189)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} x(t)$$

The weighting functions for the T-S systems are:

$$w_1(t) = \frac{\bar{\rho} - \rho(t)}{\bar{\rho} - \underline{\rho}} \quad (3.190)$$

$$w_2(t) = 1 - w_1(t) \quad (3.191)$$

The Takagi-Sugeno system can be then represented as it follows:

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=1}^2 w_i(t) (A_i x(t) + B u(t)) \\ y(t) &= C x(t) \end{aligned} \quad (3.192)$$

with

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -30 & -5 & 30 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 3.75 & 0 & -8.33 & 0 & 0 \\ 0 & 0 & 0.18 & 0 & -1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -30 & -5 & 30 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 3.75 & 0 & -2.77 & 0 & 0 \\ 0 & 0 & -0.04 & 0 & -1 \end{bmatrix}$$

### 3.3.6 Results

A generalized dynamical observer of the form:

$$\begin{aligned}
\dot{\zeta}(t) &= \sum_{i=1}^2 w_i (N_i \zeta(t) + J_i v(t) + F_i y(t) + H_i u(t)) \\
\dot{v}(t) &= \sum_{i=1}^2 w_i (S_i \zeta(t) + G_i v(t) + M_i y(t)) \\
\hat{z}(t) &= \sum_{i=1}^2 w_i (P_i \zeta(t) + Q_i y(t))
\end{aligned} \tag{3.193}$$

In order to have a comparison of the performance of the generalized dynamical observer, a proportional observer of the form

$$\begin{aligned}
\dot{\zeta}_p(t) &= \sum_{i=1}^2 w_i (N_{pi} \zeta(t) + F_{ai} y(t) + H_{pi} u(t)) \\
\hat{z}_p(t) &= \sum_{i=1}^2 w_i (P_{pi} \zeta(t) + Q_{ai} y(t))
\end{aligned} \tag{3.194}$$

is designed for the same system. The matrix  $L$  is chosen in order to estimate the hidden state of the system.

$$L = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and the matrix  $R$

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The results of the simulation are shown in Fig. 3.6. The initial conditions for the nonlinear system are  $x(0) = [\frac{\pi}{12} \ 0 \ \frac{\pi}{6} \ 0 \ 0.09]$  and the initial condition for both observers are  $\mathbf{0}$ . A parametric uncertainty of the form  $\Delta k = 0.04 \sin(t)$  is added to parameter  $k$ .

$$\begin{aligned}
\text{The matrix gains for the GDO are } N_1 = N_2 &= \begin{bmatrix} -25.62 & 7.08 & 7.14 & 7.71 \\ -24.27 & -51.32 & -0.54 & -1.79 \\ -7.4 & -1.27 & -51.49 & -2.41 \\ -9.27 & 0.55 & 1.0 & -50.92 \end{bmatrix}, \\
J_1 = J_2 &= \begin{bmatrix} 6.8 & 6.8 & 6.8 \\ 0.98 & 0.98 & 0.98 \\ 1.63 & 1.63 & 1.63 \\ 1.36 & 1.36 & 1.36 \end{bmatrix}, F_1 = \begin{bmatrix} 514.27 & 545.36 \\ -2234.93 & 149.92 \\ -67.01 & -2604.85 \\ 120.52 & 177.65 \end{bmatrix}, F_2 = \begin{bmatrix} 514.27 & 547.04 \\ -2234.93 & 149.53 \\ -67.01 & -2599.83 \\ 120.52 & 166.58 \end{bmatrix}, \\
H_1 = H_2 &= \begin{bmatrix} 0 \\ 20 \\ 0 \\ 0 \end{bmatrix}, S_1 = S_2 = \begin{bmatrix} 9.91 & 0 & 0 & 0 \\ 9.91 & 0 & 0 & 0 \\ 9.91 & 0 & 0 & 0 \end{bmatrix}, G_1 = G_2 = \begin{bmatrix} -28.77 & 3.93 & 3.93 \\ 3.93 & -28.77 & 3.93 \\ 3.93 & 3.93 & -28.77 \end{bmatrix},
\end{aligned}$$

$$M_1 = M_2 = \begin{bmatrix} -70.15 & -70.73 \\ -70.15 & -70.73 \\ -70.15 & -70.73 \end{bmatrix}, P_1 = P_2 = \begin{bmatrix} 0 & 1.0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}, Q_1 = Q_2 = \begin{bmatrix} 46.32 & 0.54 \\ 1.27 & 51.49 \\ -0.55 & -1.0 \end{bmatrix}.$$

And the matrix gains for the proportional observer are  $N_{p1} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}$ ,

$$N_{p2} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}, F_{a1} = \begin{bmatrix} -27.75 & 30.0 \\ 3.75 & -8.58 \\ 0 & 0.09 \end{bmatrix}, F_{a2} = \begin{bmatrix} -27.75 & 30.0 \\ 3.75 & -3.02 \\ 0 & -0.02 \end{bmatrix},$$

$$H_{p1} = H_{p2} = \begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix}, P_{p1} = P_{p2} = \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, Q_{a1} = Q_{a2} = \begin{bmatrix} -4.5 & 0 \\ 0 & 0.5 \\ 0 & 0 \end{bmatrix}.$$

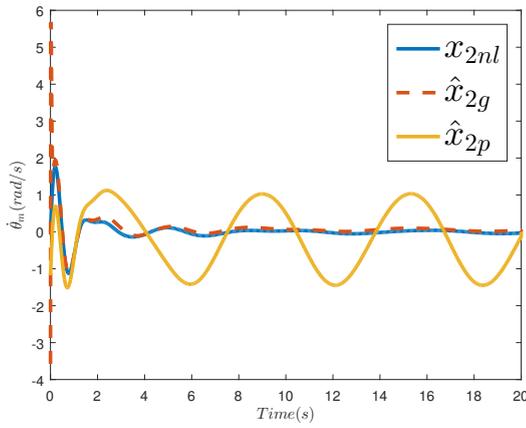
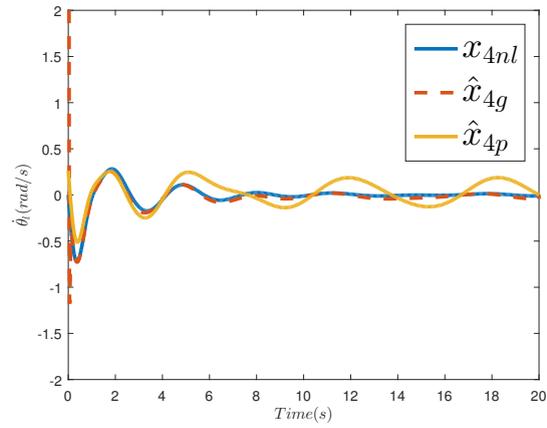
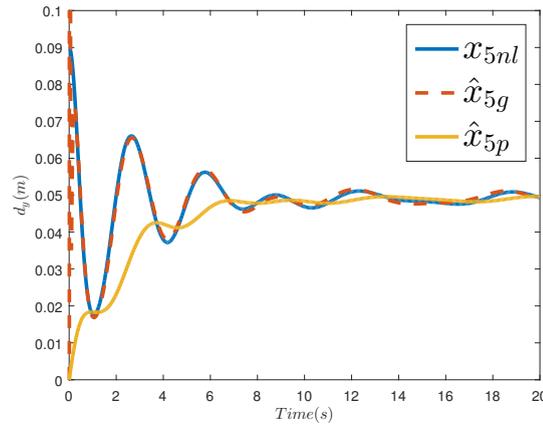
(a)  $x_2(t)$  and its estimates(b)  $x_4(t)$  and its estimates(c)  $x_5(t)$  and its estimates

Figure 3.6: Results of the simulation.

As it can be seen in Figs. 3.6, the generalized observer is capable to estimate the state of the system even in presence of parametric uncertainties. This is an advantage compared with the classical proportional observers.

### 3.3.7 Application to state and fault estimation for LTI systems

In this section we will apply the obtained results to the estimation of faults in the state equation. Consider the following descriptor system affected by faults

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + F_a f(t) \\ y(t) &= Cx(t) + F_s f(t) \\ z(t) &= Lx(t) \end{aligned} \quad (3.195)$$

where  $x(t) \in \mathbb{R}^n$  is the state of the system,  $u(t) \in \mathbb{R}^m$  is the input of the system,  $f(t) \in \mathbb{R}^{n_f}$  is the fault,  $y(t) \in \mathbb{R}^p$  is the measured output, and  $z(t) \in \mathbb{R}^q$  is a function of the states to be estimated.  $E, A, B, C, L, F_a$  and  $F_s$  are constant known matrices. The objective is to design a finite-time functional observer for simultaneous state and fault estimation for system (3.195). Before presenting the main results of this section, we can make the following remarks.

**Remark 8** *The model (3.195) is written in a general form. In fact if the actuator fault is different of the sensor one, they can be included in (3.195). Let  $f_a(t)$  and  $f_s(t)$  be the actuator and sensor faults respectively, then it is sufficient to take  $f(t) = \begin{bmatrix} f_a(t) \\ f_s(t) \end{bmatrix}$  and modify the model (3.195) according to this partition of the faults.*

**Remark 9** *Concerning the assumption on the fault  $f(t)$ , we assume that  $f(t)$  is constant or piecewise constant. This approach can also be used for time-varying faults by taking the second derivative of  $f(t)$  equal to zero and by augmenting the state of the system according to this assumption.*

By augmenting the state  $x(t)$  with the fault vector  $f(t)$ , the system (3.195) can be written as

$$\begin{aligned} \mathbf{E}\dot{\chi}(t) &= \mathbf{A}\chi(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\chi(t) \\ \check{z}(t) &= \mathbf{L}\chi(t) \end{aligned} \quad (3.196)$$

where  $\chi(t) = \begin{bmatrix} x(t) \\ f(t) \end{bmatrix}$ ,  $\check{z}(t) = \begin{bmatrix} z(t) \\ f(t) \end{bmatrix}$ ,  $\mathbf{E} = \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}$ ,  $\mathbf{A} = \begin{bmatrix} A & F_a \\ 0 & 0 \end{bmatrix}$ ,  $\mathbf{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$ ,  $\mathbf{C} = [C \ F_s]$  and  $\mathbf{L} = \begin{bmatrix} L & 0 \\ 0 & I \end{bmatrix}$ .

For this system, we can see that Assumption 1 can be written as

$$\text{rank} \begin{bmatrix} L \\ E \\ E^\perp A \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} L & 0 \\ E & 0 \\ 0 & I \\ E^\perp A & E^\perp F_a \\ C & F_a \end{bmatrix} = \text{rank} \begin{bmatrix} L \\ E \\ E^\perp A \\ C \end{bmatrix} + n_f$$

and

$$\text{rank} \begin{bmatrix} E \\ E^\perp A \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} E & 0 \\ 0 & I \\ E^\perp A & E^\perp F_a \\ C & F_a \end{bmatrix} = \text{rank} \begin{bmatrix} E \\ E^\perp A \\ C \end{bmatrix} + n_f$$

where we have the fact that  $C^+A^+ = [E^\perp A \quad E^\perp F_a]$ .

Now, the design of the observer can be achieved by using the method described above. It can be seen that, by choosing matrix  $\mathbf{L}$ , the functional observer allows to estimate the faults as well as the desired states of the augmented system.

The following algorithm can be used to obtain the different gain matrices in order to fulfill the functional observer based controller for System 3.196.

*Algorithm 1.*

1. Choose matrices  $R_i$  such that  $\text{rank} \begin{bmatrix} R_i \\ \Gamma_i \end{bmatrix} = \text{rank}(\Gamma_i)$ .
2. Compute matrices  $T_{1i}, T_{2i}, K_{1i}, K_{2i}, N_{1i}, N_{2i}$ , and  $N_{3i}$  defined in section IV.
3. Solve LMI (3.177) to obtain  $X$ , and  $Z_i$ .
4. Choose matrices  $\mathcal{L}_i$  such that  $\|\mathcal{L}_i\| < 1$ , and matrices  $\mathcal{R}_i$  such that  $\Phi_i > 0$ . Obtain  $\mathbb{Y}_i$  using (3.178).
5. Compute all the matrices of the FGDO (3.152) by using (3.160) to obtain  $T_i$ , (3.165) to obtain  $N_i$ , (3.167) to obtain  $F_i$ , (3.172)-(3.175) to obtain  $S_i, M_i, P_i$ , and  $Q_i$ , respectively and  $H_i$  by using  $H_i = T_i B$ .

## Numerical example

The generalized approach of the proposed observer is less susceptible to parametric uncertainties, compared to the proportional observer. To illustrate this, consider the following uncertain descriptor system affected by faults.

$$\begin{aligned} E\dot{x}(t) &= (A + \Delta A)x(t) + Bu(t) + F_a f(t) \\ y(t) &= Cx(t) + F_s f(t) \end{aligned} \quad (3.197)$$

$$\text{with } E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0.66 & 1.7 & 7.9 & 0.42 & -3.05 & -1.01 \\ 0.95 & -2.44 & 2.19 & 1.99 & -1.77 & 3.56 \\ -0.26 & -0.19 & -3.06 & -0.03 & 0.17 & -0.18 \\ 2.47 & 1.84 & 9.28 & -1.54 & -2.96 & -0.39 \\ 3.83 & 1.91 & 12.64 & 2.18 & -7.6 & 1.32 \\ -1.51 & -0.39 & -2.8 & -0.73 & 1.35 & -3.02 \end{bmatrix}, \Delta A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \delta(t)$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, F_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, F_s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, L = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \text{ and}$$

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The uncertainty variation  $\delta(t)$  is illustrated in Figure 3.7.

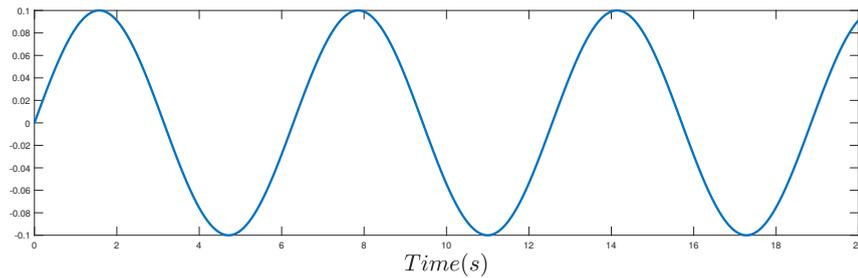


Figure 3.7: Uncertainty variation

**Simulation 3 *Finite-time Generalized Dynamic Observer performance comparison with a Proportional Observer.***

The objective of this simulation is to evaluate the performance of the Finite-time Generalized Dynamic Observer in presence of parametric uncertainties in the system and compare it to a Proportional Observer. For this system, we design a Finite-time observer based on the equations (3.152) simultaneously estimating the state and the fault, we also design the finite-time Proportional observer (see remark 2).

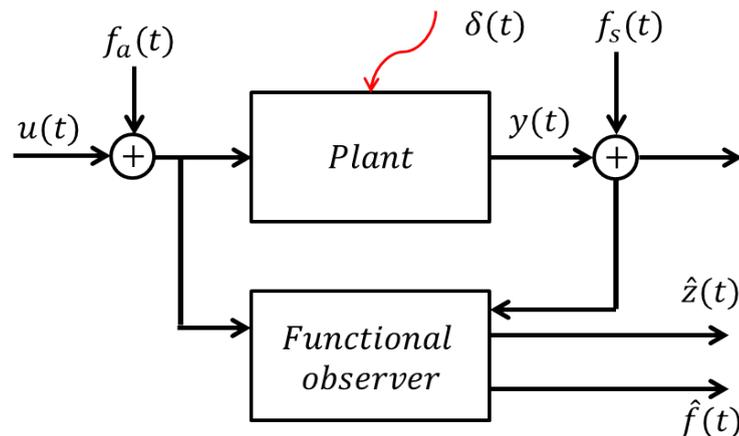


Figure 3.8: Block diagram of the functional observer

It can be seen in Figures 3.9 - 3.10, that the proposed functional observer estimates the function of the state and the fault in the given convergence time, and reduces the effects of

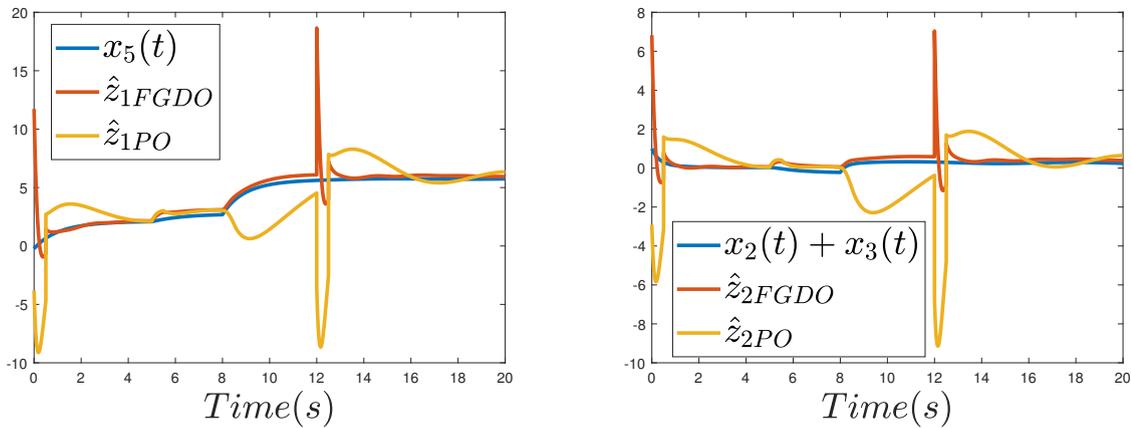


Figure 3.9: Function estimation

the parametric uncertainties in comparison to the PO. The comparison between the different observers can be seen in Table 3.2 for the Integral of the Absolute Error, and Table 3.3 for the Integral of the Square Error.

Table 3.2: IAE performance index

	$z_1(t)$	$z_2(t)$	$f_1(t)$	$f_2(t)$	$f_3(t)$
FGDO	8.33	4.88	8.35	24.21	4.81
PO	34.4	21.99	6.91	160.68	5.67

Table 3.3: ISE performance index

	$z_1(t)$	$z_2(t)$	$f_1(t)$	$f_2(t)$	$f_3(t)$
FGDO	19.35	5.15	30.39	131.86	15.92
PO	163.58	69.98	271	4 097.1	192

### 3.4 Conclusions

In conclusion, the synthesis of generalized functional observers explored in this Chapter introduces a versatile framework that enables the design of observers with dimensions distinct from the original system. This flexibility proves instrumental in crafting reduced-order observers for estimating control laws, offering practical advantages in system optimization. By incorporating

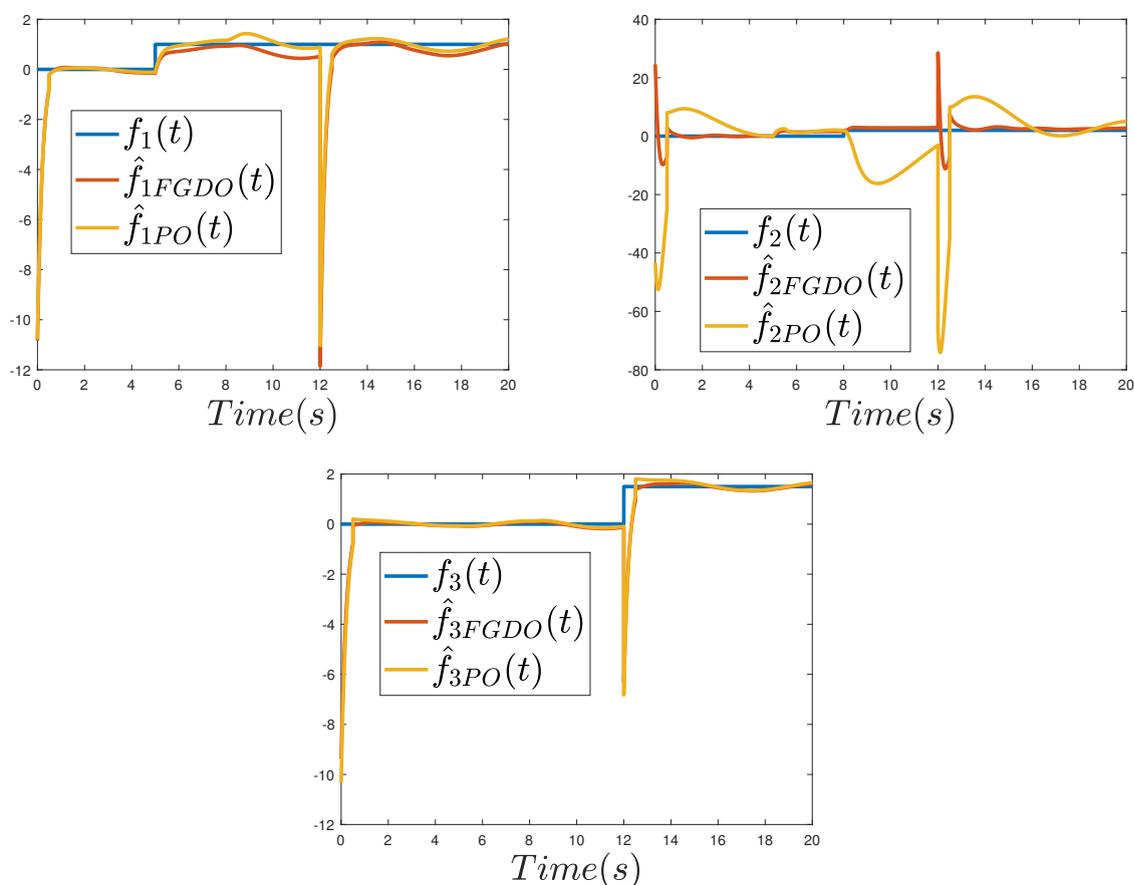


Figure 3.10: Fault estimation

augmented states that encompass faults, the methodology extends its utility to the creation of fault-tolerant control systems, particularly through the implementation of reduced-order Functional observers.

Furthermore, the generalized approach demonstrated resilience to parametric uncertainties, contrasting favorably with Proportional or Proportional-Integral observers, as evidenced in the numerical example provided. The outlined stability and existence conditions contribute to the theoretical underpinning of this approach, establishing a solid foundation for its application. The numerical example serves as a tangible illustration, showcasing the fault estimator's robust capabilities. This chapter not only advances the theoretical understanding of generalized functional observers but also highlights their practical implications for control system design and fault tolerance strategies.

In addition to its application in generalized functional observers, the synthesis approach extends seamlessly to Takagi-Sugeno modeling and linear time-invariant systems with unknown inputs. The adaptation of the framework to Takagi-Sugeno models broadens its scope, allowing for the effective handling of complex, nonlinear systems. This flexibility proves invaluable in

scenarios where traditional linear models fall short, offering a more accurate representation of real-world dynamics.

Moreover, when applied to linear time-invariant systems with unknown inputs, the synthesized observers showcase their robustness by providing accurate estimations even in the presence of uncertainties. The ability to address unknown inputs enhances the practical relevance of the methodology, especially in situations where disturbances or external influences may affect system behavior. The numerical examples presented in this chapter serve as compelling evidence of the efficacy of the proposed synthesis approach in handling the complexities associated with Takagi-Sugeno models and systems with unknown inputs. This dual extension underscores the versatility and applicability of the generalized functional observer framework across a spectrum of challenging control scenarios.

# Chapter 4

## Application of functional observers

### Contents

---

<b>4.1</b>	<b>Functional observer-based control</b>	<b>74</b>
4.1.1	Problem statement	74
4.1.2	Controller design	76
4.1.3	Numerical example	78
<b>4.2</b>	<b>Sensor fault-tolerant control</b>	<b>80</b>
4.2.1	Problem statement	80
4.2.2	Numerical example	81
<b>4.3</b>	<b>Actuator fault-tolerant control</b>	<b>83</b>
4.3.1	Problem statement	85
4.3.2	Numerical example	85
<b>4.4</b>	<b>Simultaneous state estimation and fault-tolerant control</b>	<b>88</b>
4.4.1	Problem statement	88
4.4.2	Numerical example	89
<b>4.5</b>	<b>Conclusions</b>	<b>92</b>

---

This chapter serves as a comprehensive exploration of observer-based control within the context of the generalized approach for functional observers. Observer-based control, a cornerstone of modern engineering, plays a vital role in achieving precise regulation and manipulation of dynamic processes. By accurately estimating the internal states of a system, engineers can develop control strategies that are robust, responsive, and versatile. The objective of this chapter is to provide a thorough understanding of observer-based control, its fundamental principles, design methodologies, and practical applications.

Through the exploration of functional observers, this chapter aims to provide a comprehensive understanding of their design methodologies, including the necessary and sufficient conditions for their existence.

Additionally, this chapter extends its coverage to the integration of fault-tolerant control within the framework of observer-based control. Fault tolerance ensures the system's robustness and resilience by detecting and accommodating faults that may arise during operation. The chapter explores fault detection, identification, and accommodation techniques, highlighting their integration with observer-based control to achieve simultaneous fault-tolerant control and state estimation.

## 4.1 Functional observer-based control

Let us consider the following LTI descriptor system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \\ z(t) &= Lx(t) \end{aligned} \tag{4.1}$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $u(t) \in \mathbb{R}^m$  is the input control vector,  $y(t) \in \mathbb{R}^p$  is the measured output, and  $z(t) \in \mathbb{R}^q$  is the function to estimate.  $E$ ,  $A$ ,  $B$ ,  $C$ , and  $L$  are constant known matrices.

### 4.1.1 Problem statement

The problem is to design a control law of the form  $u(t) = Lx(t) + u_r(t)$ , where  $u_r(t)$  is the reference. In this case, system 4.1 becomes:

$$\begin{aligned} E\dot{x}(t) &= (A + BL)x(t) + Bu_r(t) \\ y(t) &= Cx(t) \end{aligned} \tag{4.2}$$

Now, we can use the functional observer to estimate the functional  $z(t) = -Lx(t)$ . To find this estimate we will use the generalized observer proposed in Chapters 2 and 3.

$$\begin{aligned} \dot{\zeta}(t) &= N\zeta(t) + Jv(t) + F \begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} + Hu(t) \\ \dot{v}(t) &= S\zeta(t) + Gv(t) + M \begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} \\ \hat{z}(t) &= P\zeta(t) + Q \begin{bmatrix} -E^\perp Bu(t) \\ y(t) \end{bmatrix} \end{aligned} \tag{4.3}$$

The control system scheme can be seen in Figure 4.1.1.

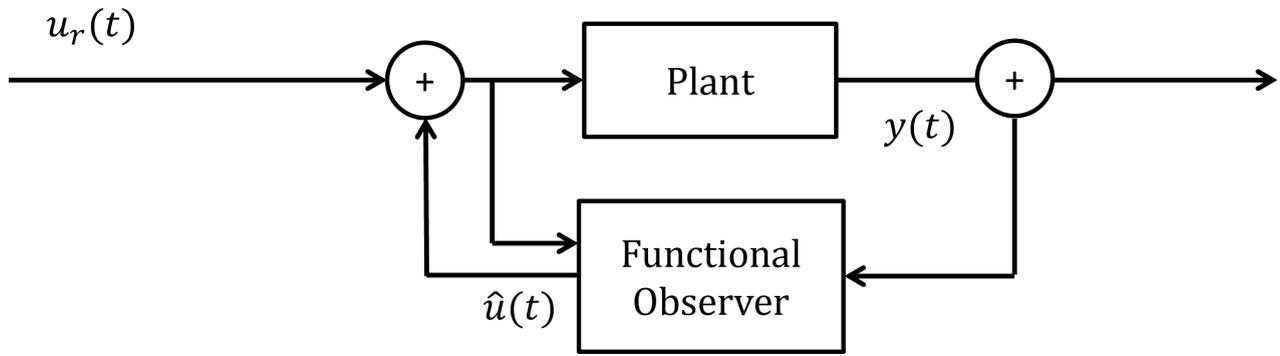


Figure 4.1: Block diagram for the observer-based control system

Then, by using the assumption that

$$\text{rank} \begin{bmatrix} L \\ E \\ E^\perp A \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} E \\ E^\perp A \\ C \end{bmatrix}$$

given in Chapter 3, and by following the approach presented in Chapter 3, we define the following transformed error.

$$\varepsilon(t) = \zeta(t) - TE x(t)$$

to obtain

$$\begin{aligned} \dot{\varepsilon}(t) &= N\varepsilon(t) + J(t) \\ \dot{v}(t) &= S\varepsilon(t) + Gv(t) \end{aligned} \tag{4.4}$$

Lets define the augmented error vector  $\sigma(t) = \begin{bmatrix} \varepsilon(t) \\ v(t) \end{bmatrix}$ , then its dynamics are represented as it follows

$$\dot{\sigma}(t) = \mathbb{A}\sigma(t) \tag{4.5}$$

where  $\mathbb{A} = \begin{bmatrix} N & J \\ S & G \end{bmatrix}$ .

Now, since  $e(t) = \hat{z}(t) - z(t) = \hat{z}(t) - Lx(t)$ , then

$$e(t) = P\varepsilon(t) \tag{4.6}$$

from (4.6), it can be seen that  $\lim_{t \rightarrow \infty} e(t) = 0$  if  $\lim_{t \rightarrow \infty} \varepsilon(t) = 0$  the stability is given when matrix  $\mathbb{A}$  is stable with the constraints

a.  $NTE + F \begin{bmatrix} E^\perp A \\ C \end{bmatrix} - TA = 0$

$$\text{b. } STE + M \begin{bmatrix} E^\perp A \\ C \end{bmatrix} = 0$$

$$\text{c. } PTE + Q \begin{bmatrix} E^\perp A \\ C \end{bmatrix} = L$$

$$\text{d. } H = TB$$

The parameters of the observer can be obtained as presented in the previous Chapters.

### 4.1.2 Controller design

Now, the controller can be obtained by combining systems (4.1) and (4.3) in this case, based on the constraints a., b., c., and d. we obtain

$$E\dot{x}(t) = Ax(t) + Bu(t) \quad (4.7)$$

$$u(t) = \hat{z}(t) + u_r(t) \quad (4.8)$$

if we substitute (4.8) into (4.7), we obtain

$$E\dot{x}(t) = Ax(t) + B\hat{z}(t) + Bu_r(t) \quad (4.9)$$

from (4.6), we have

$$\hat{z}(t) = P\varepsilon(t) + Lx(t) \quad (4.10)$$

(4.4) with (4.9) leads to

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + BP\varepsilon(t) + BLx(t) + Bu_r(t) \\ \dot{\varepsilon}(t) &= N\varepsilon(t) + J(t) \\ \dot{v}(t) &= S\varepsilon(t) + Sv(t) \end{aligned} \quad (4.11)$$

or equivalently

$$\begin{aligned} E\dot{x}(t) &= (A + BL)x(t) + BP\varepsilon(t) + Bu_r(t) \\ \dot{\varepsilon}(t) &= N\varepsilon(t) + J(t) \\ \dot{v}(t) &= S\varepsilon(t) + Sv(t) \end{aligned} \quad (4.12)$$

which can be rewritten as

$$\mathcal{E}\dot{\bar{x}}(t) = \mathcal{A}\bar{x}(t) + \mathcal{B}u_r(t) \quad (4.13)$$

$$\text{where } \bar{x}(t) = \begin{bmatrix} x(t) \\ \varepsilon(t) \\ v(t) \end{bmatrix} \text{ is an augmented vector, } \mathcal{E} = \begin{bmatrix} E & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \mathcal{A} = \begin{bmatrix} A + BL & BP & 0 \\ 0 & N & J \\ 0 & S & G \end{bmatrix}$$

$$\text{and } \mathcal{B} = \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}.$$

It can be seen from (4.13) that the system is stable if, and only if the pair  $(\mathcal{E}, \mathcal{A}, )$  is stable.

From equation (4.13), we can see that the eigenvalues of the closed loop system can be obtained from  $\mathcal{A} = 0$ , which shows that the eigenvalues of the closed loop system are the union between those of  $(E, A + BL)$  and those of  $\begin{bmatrix} N & J \\ S & G \end{bmatrix}$ .

On the other hand, the determination of the observer parameters are related to the matrix  $L$  obtained from the pair  $(E, A + BL)$ . Then, the design of the observer-based control can be obtained in two steps.

- Find a matrix  $L$ , such that the pair  $(E, A + BL)$  is stable.
- Find the matrices  $\begin{bmatrix} N & J \\ S & G \end{bmatrix}$  of the observer based on the obtained  $L$  matrix.

The following theorem gives the condition for the admissibility of (4.9).

**Theorem 4.1.1** *System (4.9) is said to be regular, impulse free and stable if and only if, there exist a matrix  $P$ , and a matrix  $X = \begin{bmatrix} X_1 & X_1 \\ X_1 & X_2 \end{bmatrix} > 0$  such that*

$$EP = P^T E^T \quad (4.14)$$

$$(A + BL)P + P^T(A + BL)^T < 0 \quad (4.15)$$

and

$$\begin{bmatrix} -X & \tilde{\mathbb{A}} \\ (*) & \mathbb{A}_1^{T\perp} X \mathbb{A}_2^{T\perp T} \end{bmatrix} < 0 \quad (4.16)$$

where  $\tilde{\mathbb{A}} = \frac{1}{\tau} \begin{bmatrix} (X_1 N_1 - W N_2 - \alpha X_1) N_3^{T\perp T} \\ (X_1 N_1 - W N_2 - \alpha X_1) N_3^{T\perp T} \end{bmatrix}$ ,  $\mathbb{A}_1 = \begin{bmatrix} N_1 - Z N_2 & 0 \\ 0 & 0 \end{bmatrix}$ , and  $\mathbb{A}_2 = \begin{bmatrix} N_3 & 0 \\ 0 & -I \end{bmatrix}$ ,

following the parameterization presented in Chapter 2, where  $\mathbb{A} = \begin{bmatrix} N & J \\ S & G \end{bmatrix}$ .

**Proof 9** *For the pair  $(E, A + LB)$  to be admissible if and only if there exists a matrix  $P_c$  such that*

$$EP_c = P_c^T E^T \geq 0 \quad (4.17)$$

and

$$(A + BL)P_c + P_c^T(A + BL)^T < 0 \quad (4.18)$$

(4.18) can be rewritten as

$$AP_c + BLP_c + P_c^T A + P_c^T L^T A^T < 0 \quad (4.19)$$

In order to avoid the bilinearity problem, let  $W_L = LP_c$ , then  $W_L^T = P_c^T L^T$ , this leads to

$$AP_c + BW_L + P_c^T A + W_L^T A^T < 0 \quad (4.20)$$

### 4.1.3 Numerical example

To illustrate this approach, we propose the following numerical example. For a system of the form (4.1), with:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -3.0 & 2.0 & 1.0 & 1.0 \\ 1.0 & -2.0 & 1.0 & -2.0 \\ 3.0 & 0 & 1.0 & -0.5 \\ -1.0 & -2.0 & 0 & 1.0 \end{bmatrix}, B = \begin{bmatrix} 1.0 & 0 \\ 0.5 & -1.0 \\ 0 & 1.0 \\ 0 & 0 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The eigenvalues of the pair  $(E, A)$  are  $\lambda = [1.79 \quad -4.4 - 1.72i \quad -4.4 + 1.72i]^T$ , from this, it can be seen that the descriptor system is unstable. In order to stabilize it, we need to find a matrix  $L$  that makes the system admissible.

A suitable Matrix is found by using the Yalmip toolbox for Matlab. In this case, the designated stabilizing matrix is

$$L = \begin{bmatrix} -8.95 & -26.07 & 12.1 & -4.85 \\ 1.71 & 1.31 & -3.51 & -0.16 \end{bmatrix}$$

and the eigenvalues of the pair  $(E, A + BL)$  are  $\lambda = [-1.7 + 1.99i \quad -1.7 - 1.99i \quad -39.78]^T$

Now we design a functional observer of the form (4.3) to estimate the function of the state  $z(t) = Lx(t)$ .

By using the method proposed in Chapter 3, we propose a generalized functional observer with the poles placed in a  $\mathcal{D}$ -region with its center in  $\alpha = -2$  and a radius  $\tau = 1$ . The matrices of the observer are

$$N = \begin{bmatrix} -3.24 & -4.26 \\ -0.1 & -2.33 \end{bmatrix}, J = \begin{bmatrix} 2.94 & 2.94 \\ 0.3 & 0.3 \end{bmatrix}, F = \begin{bmatrix} 8.12 & 39.76 & -15.39 & 17.1 \\ -2.01 & -11.6 & 5.58 & -2.45 \end{bmatrix},$$

$$H = \begin{bmatrix} -25.04 & 34.63 \\ 7.74 & -9.09 \end{bmatrix}, S = \begin{bmatrix} 0.2 & 0.7 \\ 0.11 & 0.37 \end{bmatrix}, G = \begin{bmatrix} -2.73 & -0.73 \\ -0.38 & -2.38 \end{bmatrix},$$

$$M = \begin{bmatrix} 0.05 & -0.62 & 0.76 & -0.05 \\ 0.09 & -0.26 & 0.54 & -0.09 \end{bmatrix}, P = \begin{bmatrix} 1.0 & -0.01 \\ -0.01 & 0.96 \end{bmatrix},$$

$$\text{and } Q = \begin{bmatrix} -0.37 & 4.48 & -4.29 & -4.48 \\ 1.67 & -1.53 & -0.98 & -1.83 \end{bmatrix}.$$

with these parameter matrices and controller  $L$ , the eigenvalues of the pair  $(\mathcal{E}, \mathcal{A})$  are  $[-1.7 + 1.99i \quad -1.7 - 1.99i \quad -39.78 \quad -2.0 \quad -2.0 \quad -2.2 \quad -4.47]^T$  which makes the augmented system controller-observer admissible.

### Simulation

The simulation of the Observer-based controller is done with Simulink. The initial values of the system are  $x(0) = [16 \quad -1 \quad 3 \quad 7]^T$ .  $u_r(t)$  is set to zero. The results are shown in Figures 4.1.3 and 4.1.3.

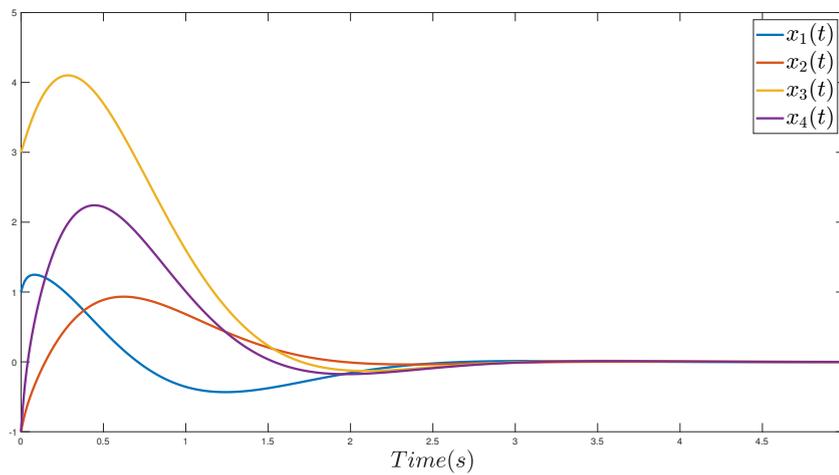


Figure 4.2: States of the system

It can be seen in Figure 4.1.3 that the states converge to 0, the observer is able to stabilize the descriptor system.

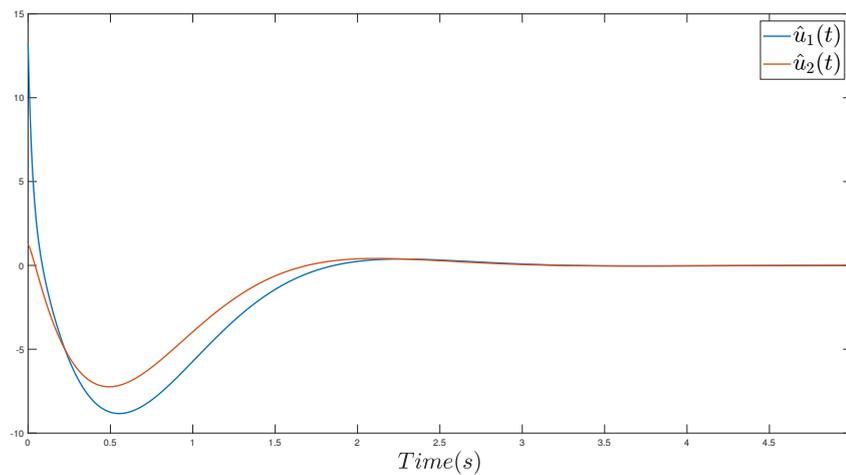


Figure 4.3: Estimated control law

A functional observer can be used as a controller for unstable descriptor LTI systems, this can be advantageous in applications where the computational cost has a great impact on the performance on the control system. For example, a functional observer can be used in small insulin pumps, where embedded systems don't have the full capability of a computer.

On the other hand, the generalized approach of the functional observer allows to mitigate the effects on parametric uncertainties to a certain extend.

## 4.2 Sensor fault-tolerant control

The FTC scheme for sensor faults is illustrated in Figure 4.4

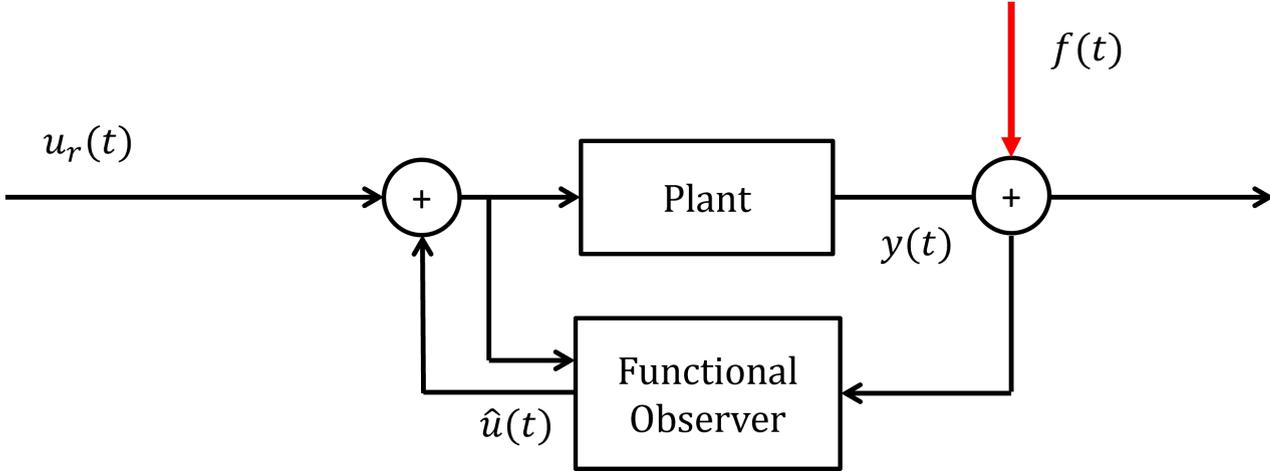


Figure 4.4: Block Diagram for sensor a FTC system

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $u(t) \in \mathbb{R}^m$  is the input control vector,  $y(t) \in \mathbb{R}^p$  is the measured output, and  $\hat{u}(t) \in \mathbb{R}^m$  is the estimate of  $u(t)$ .

### 4.2.1 Problem statement

The LTI Descriptor system is described by

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Ff(t) \\ z(t) &= Lx(t) \end{aligned} \quad (4.21)$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $F \in \mathbb{R}^{n \times n_f}$ ,  $C \in \mathbb{R}^{p \times n}$ , and  $L \in \mathbb{R}^{q \times n}$  are known constant matrices.

For the FTC we make the following assumption

**Assumption 3** *The faults in the system are constant, their dynamics are described by*

$$\dot{f}(t) = 0$$

Let  $\bar{x}(t) = [x(t) \ f(t)]^T$  be an augmented state containing both, the state of the system, and the faults. The dynamics of  $\bar{x}(t)$  are

$$\begin{aligned} \bar{E}\dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{B}\hat{u}(t) \\ y(t) &= \bar{C}\bar{x}(t) \\ \hat{z}(t) = \hat{u}(t) &= \bar{L}\bar{x}(t) \end{aligned} \quad (4.22)$$

where

$$\bar{E} = \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}, \bar{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \bar{C} = [C \quad F], \text{ and } \bar{L} = [L \quad 0]$$

In this case, the controller design follows the same process as depicted in section 4.1. Now, it can be seen that system (4.21) is stable if the pair  $(\bar{E}, \bar{A})$  is stable. This can be verified by

$$\begin{aligned} \bar{E}\bar{P}_c &= \bar{P}_c^T \bar{E}^T \geq 0 \\ (\bar{A} + \bar{B}\bar{L})\bar{P}_c + \bar{P}_c^T(\bar{A} + \bar{B}\bar{L})^T &< 0 \end{aligned} \tag{4.23}$$

## 4.2.2 Numerical example

For a system of the form (4.21) with:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -3 & 2 & 1 \\ -2 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0.5 & -1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } F_s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

we propose to design a fault-tolerant functional observer-based controller. The estimated control law is  $\hat{u}(t) = \begin{bmatrix} 2.61 & 1.09 & -1.38 \\ -0.43 & 0.04 & 0.02 \end{bmatrix} x(t)$ . Then, the augmented  $\bar{L}$  is

$$\bar{L} = \begin{bmatrix} 2.61 & 1.09 & -1.38 & 0 & 0 \\ -0.43 & 0.04 & 0.02 & 0 & 0 \end{bmatrix}$$

It can be seen that this matrix  $\bar{L}$  does not take into account the faults for the control law, the estimated control law is then  $\hat{z}(t) = Lx(t)$ , hence making the system fault-tolerant.

The functional observer of the form (4.2) has the following parameter matrices:

$$\begin{aligned} N &= \begin{bmatrix} -4.92 & 0.26 \\ 0.08 & -4.74 \end{bmatrix}, J = \begin{bmatrix} 0.22 & 0.22 \\ 0.22 & 0.22 \end{bmatrix}, F = \begin{bmatrix} 3.43 & -0.46 & -5.0 \\ -1.37 & -1.27 & 1.09 \end{bmatrix}, \\ S &= \begin{bmatrix} -0.0511 & -0.1585 \\ -0.0064 & -0.0200 \end{bmatrix}, G = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}, M = \begin{bmatrix} 0.0166 & -0.4619 & -0.339 \\ 0.0021 & -0.0582 & -0.0427 \end{bmatrix}, \\ P &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } Q = \begin{bmatrix} -0.366 & -0.1064 & -1.0110 \\ -0.1906 & -0.2696 & 0.2076 \end{bmatrix} \end{aligned}$$

## Simulation

The simulation of the Observer-based controller is done with Simulink. The initial values of the system are  $x(0) = [1 \quad -1 \quad -1]^T$ .

The sensor faults are illustrated in Figure 4.5

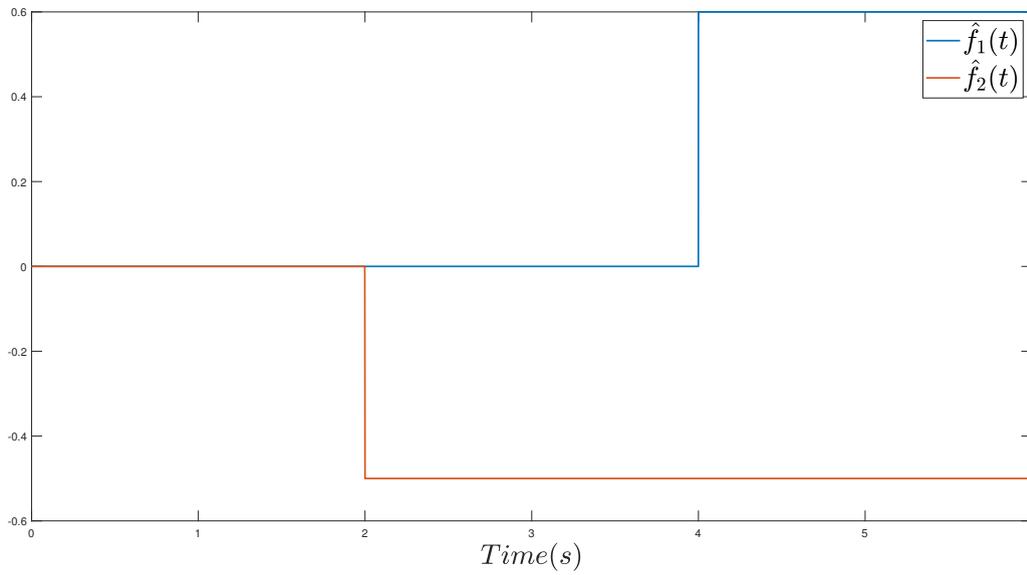


Figure 4.5: Sensor fault

The results are shown in Figures 4.6-4.8.

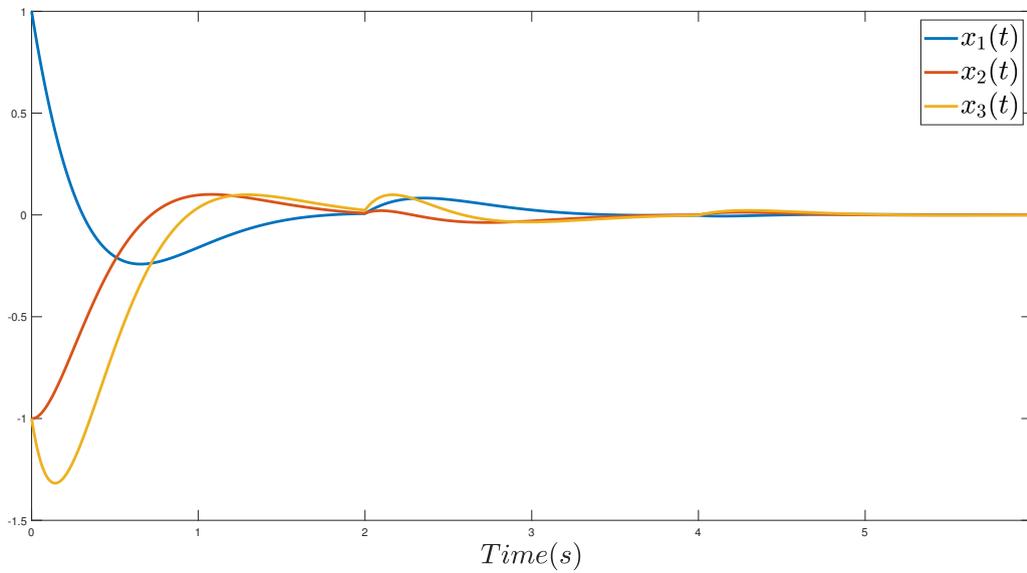


Figure 4.6: States of the system

As we can see in Figure 4.6, the states of the system are stabilized even in the presence of

the sensor faults.

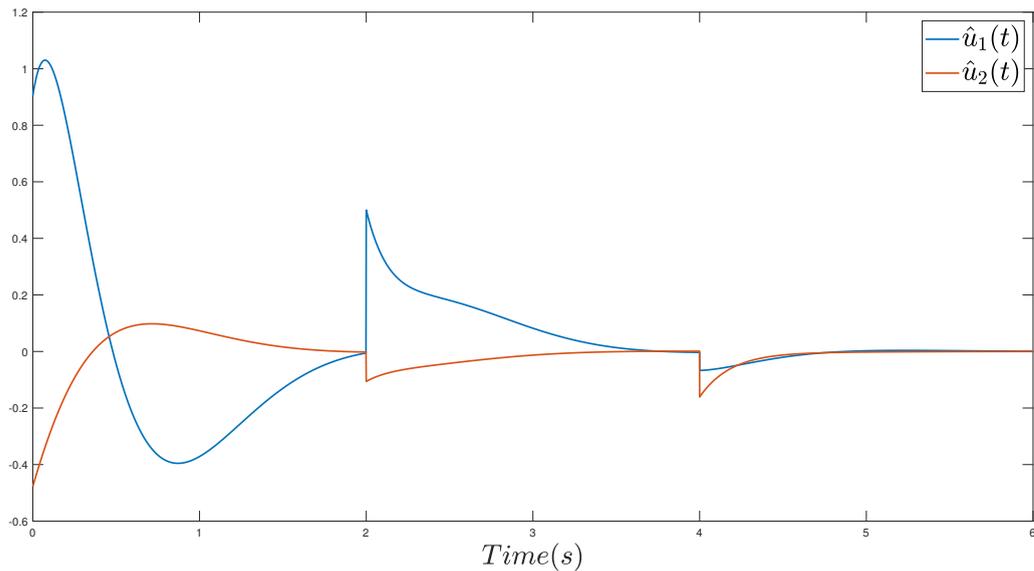


Figure 4.7: Estimated control law

The control law responds to the appearance and adapts itself to stabilize the system. In comparison, Figure 4.8 illustrate the effects of a control law  $u(t) = Lx(t)$  that is not fault-tolerant.

A functional observer can be used as a controller for descriptor LTI systems prone to sensor faults. The augmented system allows the observer to correctly estimate the function of the state while mitigating the effects of the faults in the system.

### 4.3 Actuator fault-tolerant control

The active FTC scheme used in this section is illustrated in Figure 4.9

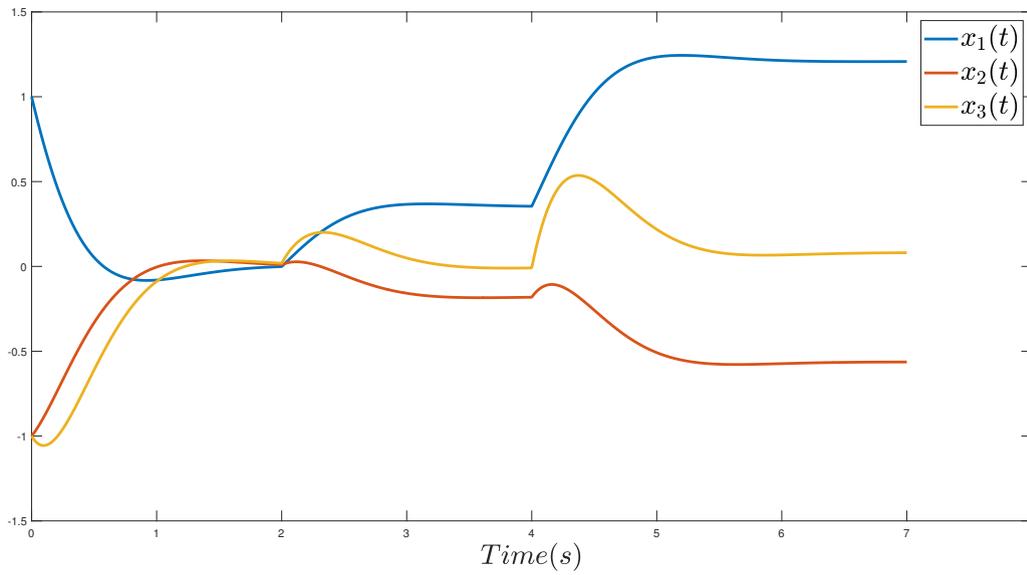


Figure 4.8: States of the system

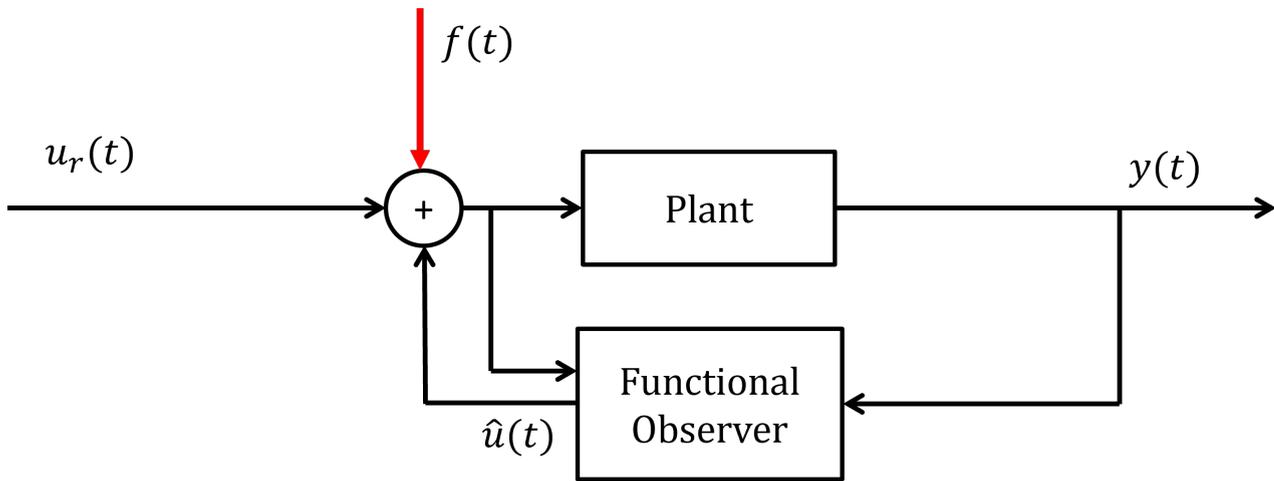


Figure 4.9: Fault-tolerant control scheme

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $u_c(t) \in \mathbb{R}^m$  is the input control vector,  $y(t) \in \mathbb{R}^p$  is the measured output, and  $\hat{u}(t) \in \mathbb{R}^m$  is the estimate of  $u_c(t)$ .

### 4.3.1 Problem statement

The LTI Descriptor system is described by

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + B(u(t) + f(t)) \\ y(t) &= Cx(t) \\ z(t) &= Lx(t) \end{aligned} \quad (4.24)$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{p \times n}$ , and  $L \in \mathbb{R}^{q \times n}$  are known constant matrices.

The FTC design is based on a functional observer that mitigates the effects of actuator faults occurring on the system. The FTC control law is given by

$$u(t) = Lx(t) - f(t) + u_r(t) \quad (4.25)$$

For the FTC design we make the following assumption

**Assumption 4** *The faults in the system are constant, their dynamics are described by*

$$\dot{f}(t) = 0$$

Let  $\bar{x}(t) = [x(t)f(t)]^T$  be an augmented state containing both, the state of the system, and the faults. The dynamics of  $\bar{x}(t)$  are

$$\begin{aligned} \bar{E}\dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{B}\hat{u}(t) \\ y(t) &= \bar{C}\bar{x}(t) \\ \hat{u}(t) &= \bar{L}\bar{x}(t) \end{aligned} \quad (4.26)$$

where

$$\bar{E} = \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}, \bar{A} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \bar{C} = [C \ 0], \text{ and } \bar{L} = [L \ -I]$$

The design of the generalized functional observer is done as stated in previous chapters, the functional  $z(t)$  must guarantee that the pair  $(\bar{E}, \bar{A})$  is stable.

### 4.3.2 Numerical example

For this example, let the system of the form (4.24) with

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -3 & 2 & 1 & 1 \\ 1 & -2 & 1 & -2 \\ 3 & 0 & 1 & -0.5 \\ -1 & -2 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0.5 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

be prone to actuator faults. The state feedback controller is calculated as

$$L = \begin{bmatrix} -8.95 & -26.07 & 12.1 & -4.85 \\ 1.71 & 1.31 & -3.51 & -0.16 \end{bmatrix}$$

For the augmented state, we have

$$\bar{L} = \begin{bmatrix} -8.95 & -267 & 12.1 & -4.85 & -1 & 0 \\ 1.71 & 1.31 & -3.51 & -0.16 & 0 & -1 \end{bmatrix}$$

In this case we chose matrix  $R$  as

$$R = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

It can be seen that the order of the observer is different from the functional  $z(t)$ . The parameter matrices are

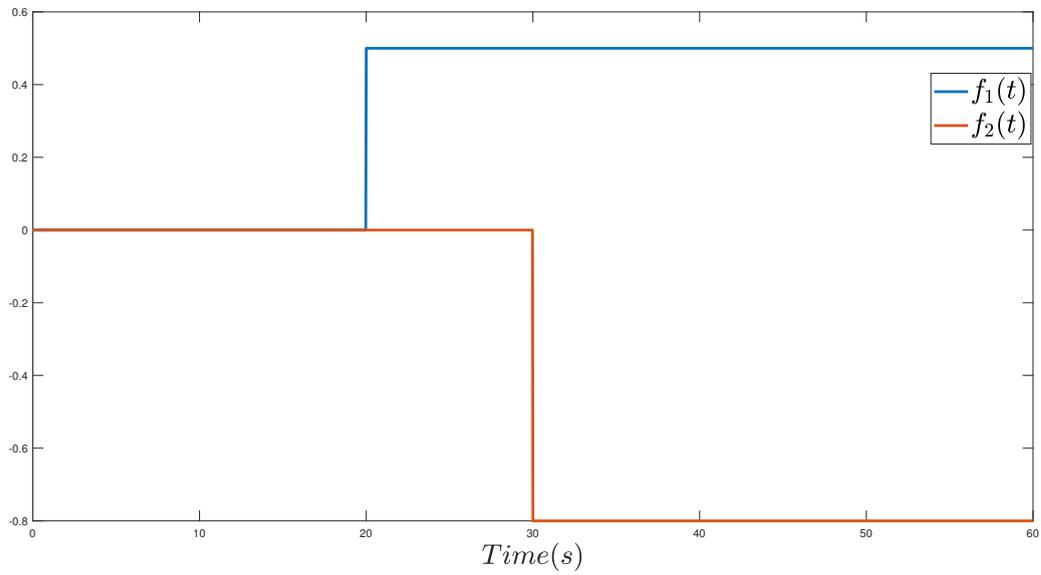
$$\begin{aligned} N &= \begin{bmatrix} 1.04 & 8.32 & 11.95 \\ -21.76 & -34.94 & -43.03 \\ 12.09 & 16.63 & 18.9 \end{bmatrix}, J = \begin{bmatrix} 0.05 & 0.05 & 0.05 \\ -0.17 & -0.17 & -0.17 \\ 0.1 & 0.1 & 0.1 \end{bmatrix}, \\ F &= \begin{bmatrix} 501.43 & 1.66 & -380.96 & 190.53 \\ -1815.274 & -48.76 & 1363.92 & -754.63 \\ 1008.47 & 18.97 & -760.63 & 405.41 \end{bmatrix}, H = \begin{bmatrix} 13 & 9.36 \\ -64.79 & -56.7 \\ 30.99 & 28.72 \end{bmatrix}, \\ S &= \begin{bmatrix} -0.0611 & -0.0278 & -0.0278 \\ -0.1110 & -0.0222 & -0.0111 \\ 0.0722 & 0.0361 & 0.0444 \end{bmatrix}, G = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}, M = \begin{bmatrix} 0.03 & 0.03 & 0.06 & -0.03 \\ 0.03 & 0.03 & 0.06 & -0.03 \\ 0.06 & 0.06 & 0.11 & -0.06 \end{bmatrix}, \\ P &= \begin{bmatrix} -12.1 & 12.1 & 11.1 \\ 2.51 & -3.51 & -2.51 \end{bmatrix}, \text{ and } Q = \begin{bmatrix} -402.34 & 432.42 & -1323.51 & 397.5 \\ 122.75 & -134.83 & 400.78 & -122.91 \end{bmatrix}. \end{aligned}$$

## Simulation

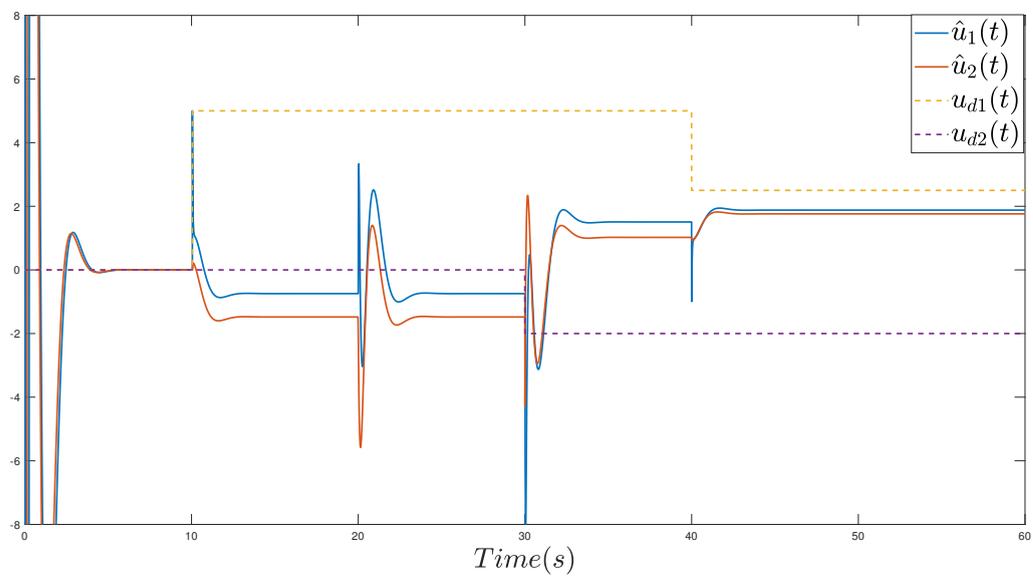
The simulation of the FTC system is done using Simulink. The initial values of the system are  $x(0) = [1 \ -1 \ 3 \ 1]^T$ . To illustrate the fault mitigation, the states of the FTC will be compared with a healthy reference model  $x_d(t)$ . Its dynamics are described by

$$\begin{aligned} E\dot{x}_d(t) &= Ax_d(t) + Bu_h(t) \\ u_h(t) &= Lx_d(t) + u_r(t) \end{aligned} \tag{4.27}$$

The actuator faults are given in Figure 4.10

Figure 4.10: Faults in the system  $f(t)$ 

The desired control law, and the comparison with the estimated fault-tolerant control law is illustrated in Figure 4.11

Figure 4.11:  $u_r(t)$  and  $\hat{u}(t)$

It can be seen that the real control law differs from the desired one. This is due to the observer compensating the controlled input in order to adjust the behavior of the system to the fault.

The results are given in Figure 4.12

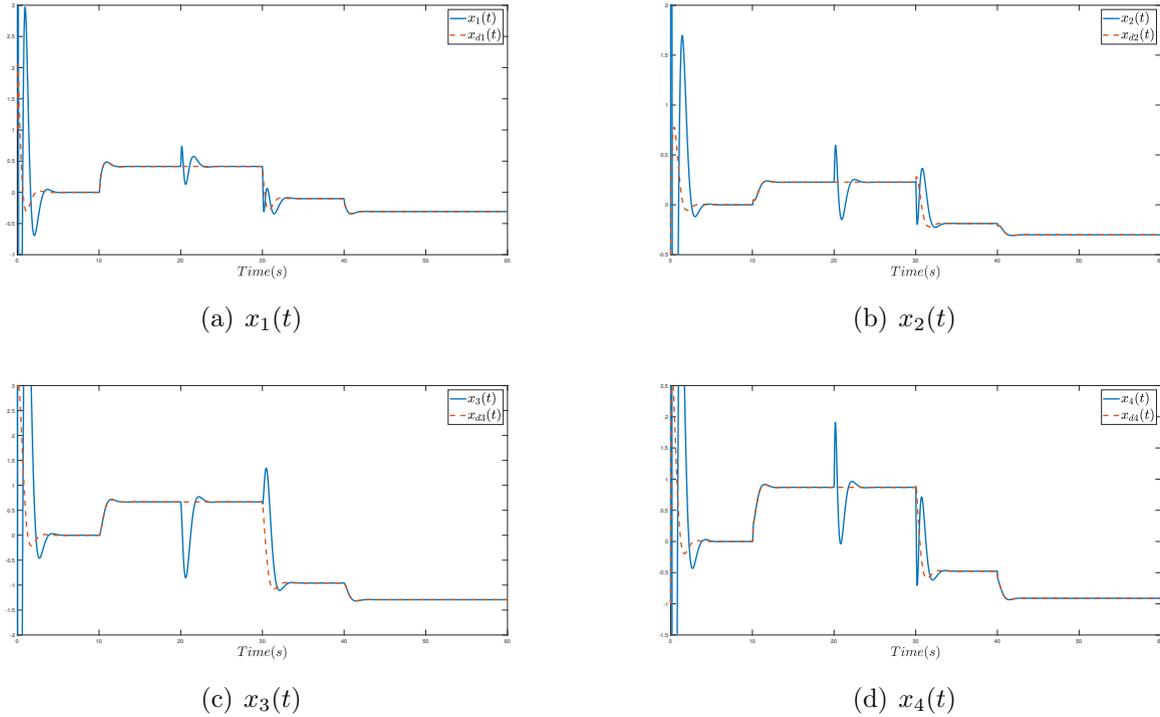


Figure 4.12: Comparison between  $x(t)$  and  $x_d(t)$

Figure 4.12 shows that the FTC mitigates the effects of the actuator faults. It can be seen that the faulty system behaves similar to a healthy reference one.

## 4.4 Simultaneous state estimation and fault-tolerant control

One of the main advantages of the functional observers is its flexibility on the estimate  $z(t)$ . This flexibility opens up possibilities for achieving simultaneous fault-tolerant stabilization, as well as fault diagnosis and isolation. By exploiting this feature, engineers can design control systems that not only ensure system stability and performance in the presence of faults but also provide valuable insights into fault detection, identification, and isolation.

### 4.4.1 Problem statement

For a descriptor LTI system of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) + F_a f(t) \\ y(t) &= Cx(t) + F_s f(t) \end{aligned} \quad (4.28)$$

where  $f(t) = [f_a(t) \ f_s(t)]^T \in \mathbb{R}^{n_f}$  is the fault vector,  $f_a(t)$  and  $f_s(t)$  are the actuator and sensor faults respectively. The stabilizing FT control law is  $u(t) = Lx(t) - f_a(t) + u_r(t)$  and will be estimated using a generalized functional observer of the form (4.3).

The first step is to design a matrix  $L$  that makes the pair  $(E, A + BL)$  stable. This method has been discussed previously in this chapter.

The system (4.28) can be rewritten as the following augmented system:

$$\begin{aligned} \bar{E}\dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{B}\hat{u}(t) \\ y(t) &= \bar{C}\bar{x}(t) \\ z(t) &= \bar{L}\bar{x}(t) \end{aligned} \quad (4.29)$$

where

$$\bar{x}(t) = \begin{bmatrix} x(t) \\ f(t) \end{bmatrix}, \bar{E} = \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}, \bar{A} = \begin{bmatrix} A & F_a \\ 0 & 0 \end{bmatrix}, \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \bar{C} = [C \ F_s], \text{ and } \bar{L} = \begin{bmatrix} L & [-I_a \ 0] \\ 0 & I_{n_f} \end{bmatrix}$$

#### 4.4.2 Numerical example

For this example, let's consider a system with sensor faults. The objective is to estimate a fault-tolerant control law able to stabilize the system as well as estimate the occurring faults. The system of the form (4.21) with.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -3.0 & 2.0 & 1.0 \\ -2.0 & 1.0 & -2.0 \\ -1.0 & -2.0 & 1.0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0.5 & -1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\text{and } F_s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The stabilizing control law is defined as  $L = \begin{bmatrix} 2.61 & 1.09 & -1.38 \\ -0.43 & 0.04 & 0.02 \end{bmatrix}$ , and the augmented  $\bar{L}$  is

$$\bar{L} = \begin{bmatrix} 2.61 & 1.09 & -1.38 & -1 & 0 \\ -0.43 & 0.04 & 0.02 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The parameter matrices are

$$N = \begin{bmatrix} -4.63 & 0.64 & 0.47 & -0.28 \\ 0.53 & -4.09 & 0.68 & -0.4 \\ 0.49 & 0.85 & -4.37 & -0.37 \\ 0.21 & 0.37 & 0.28 & -5.16 \end{bmatrix}, J = \begin{bmatrix} 0.71 & 0.71 & 0.71 & 0.71 \\ 1.01 & 1.01 & 1.01 & 1.01 \\ 0.94 & 0.94 & 0.94 & 0.94 \\ 0.41 & 0.41 & 0.41 & 0.41 \end{bmatrix},$$

$$\begin{aligned}
 F &= \begin{bmatrix} 3.43 & -0.53 & -5.05 \\ -1.37 & -1.35 & 1.04 \\ -3.41 & -3.86 & 2.27 \\ -4.32 & -10.23 & -0.45 \end{bmatrix}, \quad H = \begin{bmatrix} -2.53 & 0.36 \\ 0.52 & -0.34 \\ 1.14 & -0.91 \\ -0.23 & -1.82 \end{bmatrix} \\
 S &= \begin{bmatrix} 0.08 & 0.17 & 0.09 & -0.06 \\ -0.04 & -0.37 & 0.07 & 0.02 \\ 0 & -0.01 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}, \\
 M &= \begin{bmatrix} 0.4282 & -0.4969 & -0.3055 \\ 0.0781 & -0.5188 & -0.1316 \\ -0.7141 & 0.2415 & 0.9595 \\ 0 & 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} -0.88 & 0.16 & 0.17 & -0.09 \\ 0.16 & -0.08 & -0.09 & -0.1 \\ 0.17 & -0.09 & -0.61 & -0.14 \\ -0.09 & -0.1 & -0.14 & -0.93 \end{bmatrix}, \\
 \text{and } Q &= \begin{bmatrix} -0.37 & -0.11 & -1.01 \\ -0.19 & -0.27 & 0.21 \\ -0.45 & 0.23 & 0.45 \\ -0.91 & -2.05 & 0.91 \end{bmatrix}
 \end{aligned}$$

### Simulation

The simulation of the FTC system is done using Simulink. The initial values of the system are  $x(0) = [1 \ -1 \ -1]^T$ . The fault and its estimate is illustrated in Figure 4.13

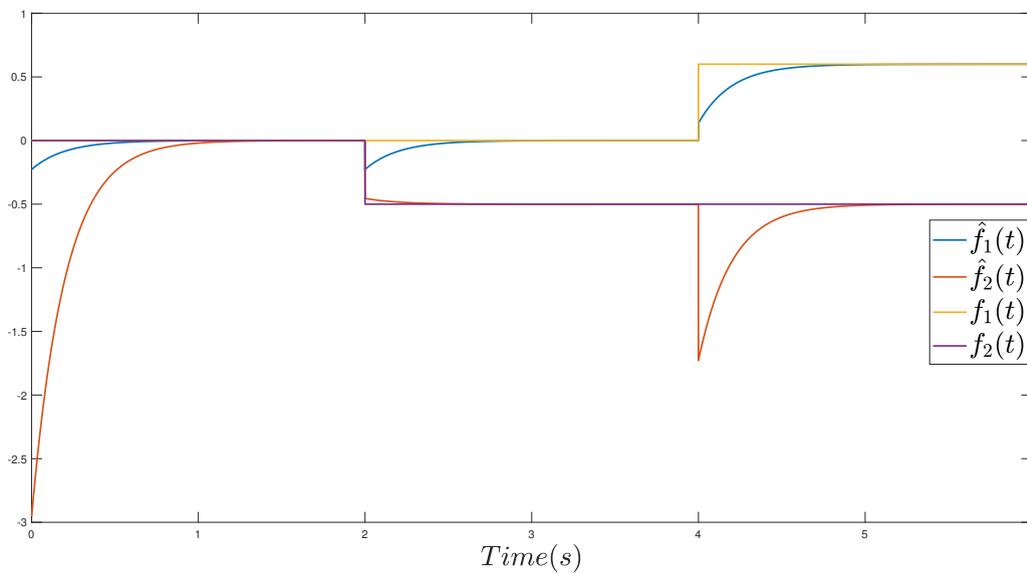


Figure 4.13: Estimated  $\hat{f}(t)$  and its comparison with  $f(t)$

It can be seen from Figure 4.13 that the faults are correctly estimated. Fault estimation

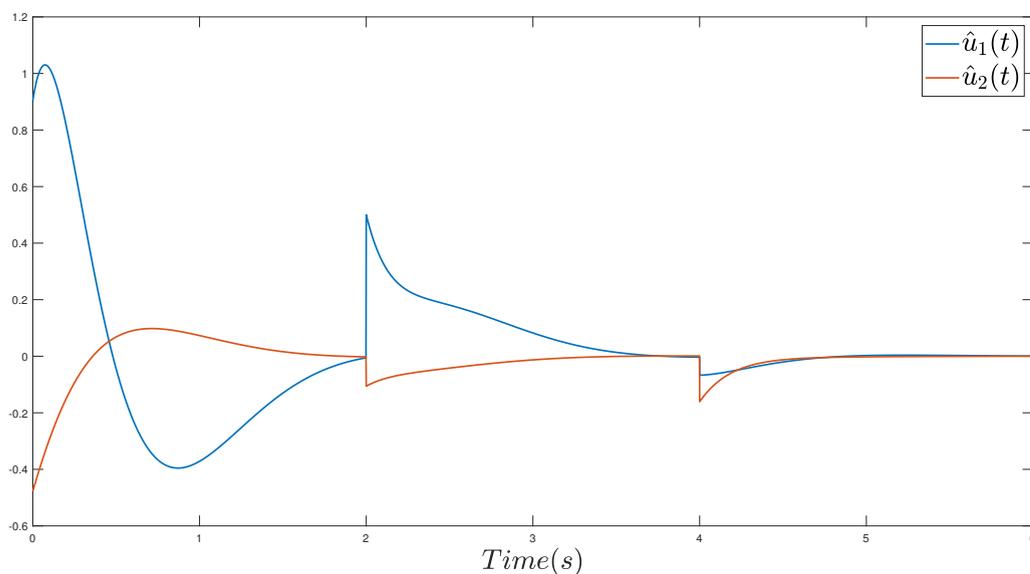


Figure 4.14: Estimated control law

contributes to improving the safety of control systems. By promptly detecting and diagnosing faults, potential safety hazards can be mitigated or avoided altogether. This is particularly critical in safety-critical applications.

Early fault detection and diagnosis facilitated by fault estimation can lead to reduced system downtime and maintenance costs. By detecting faults in their early stages, maintenance and repair activities can be scheduled proactively, minimizing unexpected system failures and reducing costly downtime. Additionally, fault estimation can aid in optimizing maintenance efforts by providing insights into the severity and urgency of the faults.

The fault-tolerant control law presented in Figure 4.14 offers an effective and robust solution for stabilizing unstable LTI descriptor systems in the presence of faults. Its integration in fault estimation techniques and ability to maintain stability and performance highlight its significance in fault-tolerant control systems.

Figure 4.15 illustrates the stable states of the system. It can be seen that the behavior of the system remains consistent and within desired bounds, indicating the effectiveness and reliability of the fault-tolerant control approach.

In conclusion, this example has presented a comprehensive investigation of a fault-tolerant control law for stabilizing an unstable LTI descriptor system in the presence of sensor faults. The proposed fault-tolerant control law has demonstrated its effectiveness in achieving system stability and maintaining desired performance objectives even in the presence of faults. Through the integration of fault estimation techniques, the control law provides timely fault detection, diagnosis, and isolation, contributing to the overall fault tolerance capabilities of the system.

The simulation results have shown that the fault-tolerant control law successfully compen-

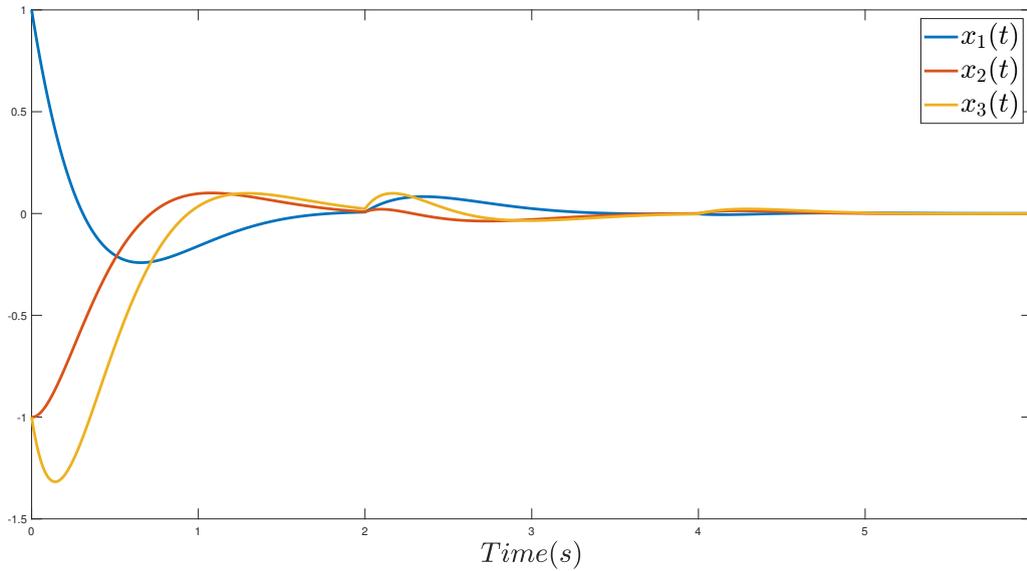


Figure 4.15: States of the system

sates for the effects of sensor faults, ensuring the stability and performance of the system. It effectively detects and isolates faults, allowing for appropriate corrective actions to be taken in a timely manner. The control law has exhibited robustness and resilience against different fault scenarios, accommodating variations in fault magnitudes, durations, and patterns.

## 4.5 Conclusions

This chapter has provided a comprehensive exploration of observer-based control within the context of functional observers for linear descriptor systems. The fundamental principles, design methodologies, and practical applications of observer-based control have been thoroughly discussed, highlighting its crucial role in achieving precise regulation and manipulation of dynamic processes.

Through the exploration of functional observers, we have gained a comprehensive understanding of their design methodologies, including the necessary and sufficient conditions for their existence. The insights gained from this exploration contribute to the body of knowledge on observer-based control, empowering engineers to design state estimators that accurately estimate the internal states of a system, leading to the development of robust, responsive, and versatile control strategies.

Moreover, the integration of fault-tolerant control within the framework of observer-based control has been addressed. Fault tolerance is essential for ensuring system robustness and resilience in the face of faults that may arise during operation. By incorporating fault detection, identification, and accommodation techniques with observer-based control, simultaneous fault-

tolerant control and state estimation can be achieved, enhancing the overall performance and reliability of control systems.

The case studies and simulations presented in this chapter have demonstrated the effectiveness of the proposed observer-based fault-tolerant control approach for linear descriptor systems. The results have showcased improved system performance, robustness against faults, and the ability to maintain accurate state estimation even in the presence of faults or uncertainties.

By leveraging functional observers, engineers can design control systems that are not only robust and resilient to faults but also capable of accurately estimating system states, leading to improved performance and reliability. The advancements made in this area have the potential to positively impact various industries, including aerospace, automotive, robotics, and process control, among others.

As we move forward, it is essential to continue exploring, refining, and expanding upon the concepts and techniques discussed in this chapter. By doing so, we can advance the state-of-the-art in observer-based control and fault-tolerant control, bringing us closer to achieving more reliable and robust control systems in the face of uncertainties and faults.



# Chapter 5

## Conclusions

### Contents

---

<b>5.1</b>	<b>Contributions . . . . .</b>	<b>96</b>
<b>5.2</b>	<b>Products . . . . .</b>	<b>97</b>
<b>5.3</b>	<b>Discussion . . . . .</b>	<b>97</b>
<b>5.4</b>	<b>Limitations . . . . .</b>	<b>98</b>
<b>5.5</b>	<b>Future works . . . . .</b>	<b>99</b>

---

The work presented in this thesis contributes to the estimation, control, and fault diagnosis for Linear Descriptor Systems. Functional observers have a good potential in control theory, their simplicity allows us to design fast and reliable control systems and use them in combination with other tools to create powerful diagnostic and fault-tolerant systems.

In Chapter 3 a method for the synthesis of Finite-Time Functional Observers for continuous-time descriptor systems was presented. The existence conditions were given and the stability conditions were set in terms of LMI approach. D-stability was chosen to guarantee the finite-time convergence of the observer. This Chapter also explores the non-linear systems and proposes a method for the synthesis of a Takagi-Sugeno functional observer. In this chapter we concluded that the linear theory of the functional observers proposed can be used to estimate the state of Takagi-Sugeno systems. This TS systems can be used to represent non-linear behaviors of a plant. Finally, in Chapter 3, unknown inputs are presented in linear systems, and are estimated by using a proposed functional observer.

In Chapter 4, the focus is on the problem of observer-based control for descriptor linear time-invariant (LTI) systems. The chapter delves into the synthesis of fault-tolerant control strategies specifically designed to address sensor faults, actuator faults, and the simultaneous combination of fault-tolerant control and fault estimation. The developed fault-tolerant control methodologies offer effective solutions for mitigating the effects of sensor faults, actuator faults, and the simultaneous occurrence of faults while ensuring the stability and desired performance of the system. The integration of fault estimation techniques within the control framework

contributes to the fault tolerance capabilities, enabling timely detection, diagnosis, and accommodation of faults.

In this final chapter, we summarize the key findings, contributions, and implications of the research conducted in this thesis. We revisit the main objectives outlined in the introduction and reflect on the insights gained throughout the study. Additionally, we discuss the limitations of the research and propose avenues for future research in the field.

## 5.1 Contributions

This research has made significant contributions to the field of functional observers and control for descriptor systems. The following key contributions have been achieved:

1. Development of Finite-Time Convergence Generalized Functional Observers:
  - We have proposed and investigated the novel concept of finite-time convergence generalized functional observers.
  - This contribution extends the traditional functional observer framework by ensuring convergence within a finite time horizon, providing a valuable tool for time-critical applications.
2. Functional Observers for Takagi-Sugeno Descriptor Systems:
  - We have developed functional observers specifically tailored for Takagi-Sugeno descriptor systems, which are widely used in modeling complex systems with nonlinear dynamics.
  - The proposed observers offer a systematic and effective approach for state estimation in Takagi-Sugeno descriptor systems, considering both continuous and discrete modes.
3. Generalized Dynamical Observer with Unknown Inputs:
  - We have introduced the concept of the generalized dynamical observer with unknown inputs, which enables the estimation of both system states and unknown inputs.
  - This contribution enhances the observer's capability to handle unmeasured inputs, improving the accuracy of state estimation and fault detection in practical systems.
4. Functional Observer-Based Control for Descriptor Systems:
  - We have explored the integration of functional observers into the control framework for descriptor systems.
  - By combining state estimation and control in a unified framework, our research provides a comprehensive solution for achieving robust and efficient control in the presence of uncertainties and disturbances.

These contributions advance the state-of-the-art in functional observers and control for descriptor systems, offering novel methodologies and techniques to address challenging real-world problems. The proposed approaches extend the existing theoretical foundations and provide practical solutions that have potential applications in diverse fields.

Moreover, the contributions of this research offer practical implications for system design, fault diagnosis, and fault-tolerant control strategies. The developed methodologies enhance system performance, reliability, and fault tolerance, leading to improved operational efficiency and reduced downtime in critical applications.

## 5.2 Products

As results for this research study, four scientific articles have been accepted and presented; two in national congresses, with titled “**Generalized Functional Observer Design for Descriptor Linear Systems**” (In Congreso Nacional de Control Automático 2017, October 2017, Monterrey, Mexico) and “**Synthèse d’observateurs fonctionnels à temps de convergence finie**” (In Congrès National des IUT, June 2019, Toulon, France).

And two articles in international congresses, titled “**Finite time functional observers for descriptor systems. Application to fault tolerant control**”. (In 2019 27<sup>th</sup> Mediterranean Conference on Control and Automation (MED)), and “**Design of finite-time convergent functional observers**” (In 2019 8<sup>th</sup> International Conference on Systems and Control, October 2019, Marrakesh, Morocco).

The presented results were used to publish a scientific article in the magazine *Processes* (doi: 10.3390/pr11061707) with the title **Generalized Functional Observer for Descriptor Nonlinear Systems—A Takagi-Sugeno Approach**.

## 5.3 Discussion

In this section, we engage in a comprehensive discussion of the research findings, their significance, and their implications for the field of functional observers and control for descriptor systems. We also address any limitations and challenges encountered during the research process.

The key findings of this research significantly contribute to the field of functional observers and control for descriptor systems. The development of finite-time convergence generalized functional observers ensures convergence within a finite time horizon, addressing time-critical applications and advancing the field’s knowledge. The tailored functional observers for Takagi-Sugeno descriptor systems enable state estimation in complex systems with nonlinear dynamics, enhancing the applicability of functional observers in practical engineering domains. The integration of functional observers into the control framework for descriptor systems provides a unified approach that combines state estimation and control, leading to improved performance, fault tolerance, and adaptability. These findings align with the original research objectives outlined in the introduction and offer novel methodologies, enhanced performance, and practical

implications for real-world applications. Overall, these key findings contribute to the advancement of knowledge and understanding in the field, extending the capabilities of functional observers and control strategies for descriptor systems.

The research findings have significant implications for real-world applications in various industrial systems and engineering domains. The developed methodologies and techniques offer practical solutions for improving system performance, reliability, and fault tolerance in practical scenarios.

The implementation of finite-time convergence generalized functional observers in time-critical applications can have a profound impact. Industries such as aerospace, robotics, and autonomous vehicles can benefit from the real-time estimation capabilities of these observers, enabling faster and more accurate decision-making. The ability to ensure convergence within a finite time horizon enhances system stability, safety, and operational efficiency in time-constrained scenarios.

The tailored functional observers for Takagi-Sugeno descriptor systems have direct implications for industries dealing with complex systems exhibiting nonlinear dynamics. By accurately estimating system states, these observers facilitate system monitoring, fault diagnosis, and control in domains such as process industries, power systems, and chemical engineering. The robust estimation capabilities of these observers enable precise modeling and control, improving system performance and reliability.

The integration of functional observers into the control framework for descriptor systems offers practical benefits for enhancing system performance, reliability, and fault tolerance. Industries with safety-critical applications, such as automotive systems, industrial automation, and renewable energy systems, can leverage these techniques to ensure uninterrupted operation and improve system resilience. By combining state estimation and control, these approaches enable proactive fault management and adaptive control actions, mitigating the effects of faults and uncertainties and enhancing overall system reliability.

Overall, the research findings have practical implications across a range of industrial systems and engineering domains. The developed methodologies and techniques provide practical solutions to improve system performance, reliability, and fault tolerance. Their implementation can lead to enhanced operational efficiency, reduced downtime, improved safety, and increased system resilience in real-world applications.

## 5.4 Limitations

It is important to acknowledge the limitations and constraints of this work. These limitations provide opportunities for future research and areas of improvement.

A limitation of this work is its reliance on simplified models and assumptions, which were necessary to facilitate the development of control strategies and observer designs. Certain assumptions were made regarding system dynamics, noise characteristics, and fault patterns to enable theoretical advancements. However, it is important to acknowledge that these simplified models and assumptions may not fully capture the complexity and variability of real-world systems. While they provided a foundation for theoretical developments, future research should

focus on considering more realistic and complex models to enhance the practical applicability of the developed methodologies. Incorporating more intricate and diverse system dynamics, as well as realistic noise characteristics and fault patterns, will allow for a more comprehensive assessment of the proposed approaches and ensure their effectiveness in real-world applications.

A further consideration lies in the practical implementation of the proposed methodologies and their robustness to uncertainties. Real-world systems commonly exhibit unknown or time-varying parameters, uncertainties, and disturbances that can significantly impact the performance of functional observers and control strategies. While the proposed methodologies show promise in controlled environments, their practical implementation may face additional challenges. To overcome this limitation, future research should prioritize the development of robust methodologies that can effectively handle uncertainties, adapt to changing conditions, and ensure reliable system operation under realistic scenarios. By incorporating robustness considerations and accounting for uncertainties, the proposed methodologies can enhance their practical applicability and provide more reliable and resilient control solutions for real-world systems.

An important consideration is the limited focus on experimental validation and real-world case studies in this work. While the research has primarily concentrated on theoretical developments and simulation-based evaluations, it is crucial to supplement these findings with experimental validation to fully assess the practical effectiveness and robustness of the proposed methodologies. Conducting experiments in realistic environments and collaborating with industry partners can provide valuable insights and validation of the developed techniques. By incorporating experimental validation and real-world case studies, researchers can gain a deeper understanding of the practical implications and limitations of the proposed methodologies, ensuring their applicability in real-world scenarios and validating their performance under varying conditions. Further emphasis on experimental validation and real-world case studies will enhance the credibility and practical relevance of the research outcomes.

This research has addressed key challenges and proposed methodologies to enhance the resilience, performance, and practical applicability of these systems. However, certain limitations, such as the reliance on simplified models and assumptions, the need for robustness to uncertainties, and the importance of experimental validation and real-world case studies, have been identified. Future research should focus on addressing these limitations and further advancing the proposed methodologies. By considering more realistic and complex models, developing robust methodologies, and conducting experimental validations, the proposed approaches can be refined and their effectiveness in real-world scenarios can be thoroughly assessed. These efforts will contribute to the development of more reliable, adaptable, and practical solutions for functional observers and control in descriptor systems.

## 5.5 Future works

While this research has made significant contributions to the field of functional observers and control for descriptor systems, several promising avenues for future research and development emerge. Specifically, future works should focus on addressing the challenges posed by actuator

and sensor saturation, as well as extending the proposed methodologies to handle nonlinear systems, and considering the effects of noise in output measurements.

### 1. Actuator and Sensor Saturation:

- Future research should focus on incorporating actuator and sensor saturation into the control framework for descriptor systems. Considering the impact of saturation on system stability and performance is crucial for practical applications. Developing control strategies that explicitly account for actuator saturation and propose appropriate saturation mitigation techniques will be valuable.
- Similarly, addressing the challenges posed by sensor saturation is important for reliable state estimation and fault detection. Investigating robust observer-based approaches that can handle sensor saturation and mitigate its effects on state estimation accuracy will improve the overall system's performance.

### 2. Extension to Nonlinear Systems:

- Extending the proposed methodologies to handle nonlinear systems is an important direction for future research. Many real-world systems exhibit nonlinear behavior, and incorporating nonlinear dynamics into the observer design and control strategies is essential.
- Investigating adaptive functional observers or nonlinear observer-based control methods can offer promising solutions for nonlinear systems. These techniques can adaptively estimate the nonlinear dynamics and handle uncertainties, improving the accuracy and robustness of state estimation and control.

### 3. Handling the Effects of Noise in Output Measurement:

- In practical systems, output measurements are often corrupted by noise, which can impact the accuracy of state estimation and fault detection. Future research should focus on developing robust methodologies that explicitly account for the effects of noise in output measurements.
- Investigating noise-robust observer design techniques, such as filtering or estimation algorithms that can effectively mitigate the impact of noise, will enhance the accuracy and reliability of state estimation.

### 4. Validation and Real-World Implementation:

- Validating and implementing the proposed methodologies in real-world scenarios is essential. Conducting extensive experiments, case studies, or industrial collaborations will provide valuable insights into the performance and practical applicability of the developed techniques, considering the effects of measurement noise.

- Additionally, investigating the scalability and adaptability of the proposed methodologies to large-scale systems or different application domains, while accounting for measurement noise, will be crucial for their wider adoption and successful implementation.

By addressing the challenges of actuator and sensor saturation, extending the methodologies to handle nonlinear systems, integrating fault diagnosis and fault-tolerant control while considering the effects of noise in output measurements, and validating the approaches in real-world scenarios, future research can further enhance the capabilities and practical applicability of functional observers and control for descriptor systems. These areas offer exciting opportunities for advancing the field and addressing the complex challenges in system control, fault tolerance, and the presence of measurement noise.



# Bibliography

- [Aldeen and Trinh, 1999] Aldeen, M. and Trinh, H. (1999). Reduced-order linear functional observer for linear systems. *IEE Proceedings-Control Theory and Applications*, 146(5):399–405.
- [Ali et al., 2015] Ali, J. M., Hoang, N. H., Hussain, M. A., and Dochain, D. (2015). Review and classification of recent observers applied in chemical process systems. *Computers & Chemical Engineering*, 76:27–41.
- [Araujo et al., 2012] Araujo, J. M., Barros, P. R., and Dorea, C. E. (2012). Design of observers with error limitation in discrete-time descriptor systems: A case study of a hydraulic tank system. *IEEE Transactions on Control Systems Technology*, 20(4):1041–1047.
- [Chakrabarty et al., 2017] Chakrabarty, A., Corless, M. J., Buzzard, G. T., Žak, S. H., and Rundell, A. E. (2017). State and unknown input observers for nonlinear systems with bounded exogenous inputs. *IEEE Transactions on Automatic Control*, 62(11):5497–5510.
- [Chilali and Gahinet, 1996] Chilali, M. and Gahinet, P. (1996). H/sub/spl infin//design with pole placement constraints: an lmi approach. *IEEE Transactions on automatic control*, 41(3):358–367.
- [Cobb, 1984] Cobb, D. (1984). Controllability, observability, and duality in singular systems. *IEEE Transactions on Automatic Control*, 29(12):1076–1082.
- [Dai, 1989] Dai, L. (1989). *Singular control systems (Lecture notes in control and information sciences)*. Springer.
- [Darouach, 2000] Darouach, M. (2000). Existence and design of functional observers for linear systems. *IEEE Transactions on Automatic Control*, 45(5):940–943.
- [Darouach, 2001] Darouach, M. (2001). Linear functional observers for systems with delays in state variables. *IEEE Transactions on Automatic Control*, 46(3):491–496.
- [Darouach, 2012] Darouach, M. (2012). On the functional observers for linear descriptor systems. *Systems and Control Letters*, 61(3):427–434.
- [Darouach and Boutayeb, 1995] Darouach, M. and Boutayeb, M. (1995). Design of observers for descriptor systems. *IEEE transactions on Automatic Control*, 40(7):1323–1327.

- [Darouach et al., 1996] Darouach, M., Zasadzinski, M., and Hayar, M. (1996). Reduced-order observer design for descriptor systems with unknown inputs. *IEEE transactions on automatic control*, 41(7):1068–1072.
- [Davila et al., 2005] Davila, J., Fridman, L., and Levant, A. (2005). Second-order sliding-mode observer for mechanical systems. *IEEE transactions on automatic control*, 50(11):1785–1789.
- [Dodig and Stošić, 2009] Dodig, M. and Stošić, M. (2009). Singular systems, state feedback problem. *Linear Algebra and its Applications*, 431(8):1267–1292.
- [Duan, 2010] Duan, G.-R. (2010). *Analysis and design of descriptor linear systems*, volume 23. Springer Science & Business Media.
- [Engel and Kreisselmeier, 2002] Engel, R. and Kreisselmeier, G. (2002). A continuous-time observer which converges in finite time. *IEEE Transactions on Automatic Control*, 47(7):1202–1204.
- [Fernando et al., 2011] Fernando, T. L., Jennings, L. S., and Trinh, H. M. (2011). Generality of functional observer structures. In *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*, pages 4000–4004. IEEE.
- [Fernando et al., 2010] Fernando, T. L., Trinh, H. M., and Jennings, L. (2010). Functional observability and the design of minimum order linear functional observers. *IEEE Transactions on Automatic Control*, 55(5):1268–1273.
- [Garcia and Bernussou, 1995] Garcia, G. and Bernussou, J. (1995). Pole assignment for uncertain systems in a specified disk by state feedback. *IEEE Transactions on Automatic Control*, 40(1):184–190.
- [Gu and Yang, 2017] Gu, Y. and Yang, G.-H. (2017). Sensor fault estimation for lipschitz nonlinear systems in finite-frequency domain. *International Journal of Systems Science*, pages 1–11.
- [Guelton et al., 2008] Guelton, K., Delprat, S., and Guerra, T.-M. (2008). An alternative to inverse dynamics joint torques estimation in human stance based on a takagi–sugeno unknown-inputs observer in the descriptor form. *Control Engineering Practice*, 16(12):1414–1426.
- [Kučera and Zagalak, 1988] Kučera, V. and Zagalak, P. (1988). Fundamental theorem of state feedback for singular systems. *Automatica*, 24(5):653–658.
- [Lan and Patton, 2015] Lan, J. and Patton, R. J. (2015). Robust fault-tolerant control based on a functional observer for linear descriptor systems. *IFAC-PapersOnLine*, 48(14):138–143.
- [Lewis, 1986] Lewis, F. L. (1986). A survey of linear singular systems. *Circuits, Systems, and Signal Processing*, 5(1):3–36.

- [Luenberger, 1966] Luenberger, D. (1966). Observers for multivariable systems. *IEEE Transactions on Automatic Control*, 11(2):190–197.
- [Luenberger, 1971] Luenberger, D. (1971). An introduction to observers. *IEEE Transactions on automatic control*, 16(6):596–602.
- [Luenberger, 1978] Luenberger, D. G. (1978). Time-invariant descriptor systems. *Automatica*, 14(5):473–480.
- [Müller and Hou, 1991] Müller, P. and Hou, M. (1991). On the observer design for descriptor systems. In *Decision and Control, 1991., Proceedings of the 30th IEEE Conference on*, pages 1960–1961. IEEE.
- [Osorio-Gordillo et al., 2015] Osorio-Gordillo, G.-L., Darouach, M., and Astorga-Zaragoza, C.-M. (2015).  $H_\infty$  dynamical observers design for linear descriptor systems. application to state and unknown input estimation. *European Journal of Control*, 26:35–43.
- [Osorio-Gordillo et al., 2016] Osorio-Gordillo, G. L., Darouach, M., Astorga-Zaragoza, C. M., and Boutat-Baddas, L. (2016). New dynamical observers design for linear descriptor systems. *IET Control Theory & Applications*, 10(17):2223–2232.
- [Perruquetti et al., 2008] Perruquetti, W., Floquet, T., and Moulay, E. (2008). Finite-time observers: application to secure communication. *IEEE Transactions on Automatic Control*, 53(1):356–360.
- [Shafai and Carroll, 1985] Shafai, B. and Carroll, R. (1985). Design of proportional-integral observer for linear time-varying multivariable systems. In *Decision and Control, 1985 24th IEEE Conference on*, volume 24, pages 597–599. IEEE.
- [Shafai et al., 2002] Shafai, B., Pi, C., Nork, S., and Linder, S. (2002). Proportional integral adaptive observer for parameter and disturbance estimations. In *Decision and Control, 2002, Proceedings of the 41st IEEE Conference on*, volume 4, pages 4694–4699. IEEE.
- [Shen and Xia, 2008] Shen, Y. and Xia, X. (2008). Semi-global finite-time observers for nonlinear systems. *Automatica*, 44(12):3152–3156.
- [Skelton et al., 1997] Skelton, R. E., Iwasaki, T., and Grigoriadis, D. E. (1997). *A unified algebraic approach to control design*. CRC Press.
- [Trinh and Fernando, 2011] Trinh, H. and Fernando, T. (2011). *Functional observers for dynamical systems*, volume 420. Springer Science & Business Media.
- [Yao et al., 1997] Yao, Y., Zhang, Y., and Kovacevic, R. (1997). Functional observer and state feedback for input time-delay systems. *International Journal of control*, 66(4):603–617.
- [Zhang, 2006] Zhang, G. (2006). Regularizability, controllability and observability of rectangular descriptor systems by dynamic compensation. In *American Control Conference, 2006*, pages 6–pp. IEEE.

[Zhang et al., 2016] Zhang, K., Jiang, B., and Cocquempot, V. (2016). Robust fault estimation observer design with finite-time convergence specification. *IET Control Theory & Applications*, 11(1):1–9.

[Zhang, 1987] Zhang, S.-Y. (1987). Functional observer and state feedback. *International Journal of Control*, 46(4):1295–1305.