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Clara Dérand

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# Positivité, hyperbolicité et théorie de Nevanlinna

## THÈSE

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pour l'obtention du

**Doctorat de l'Université de Lorraine**

**Mention mathématiques**

par

Clara Dérand

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## Résumé

Cette thèse s'inscrit dans le domaine de l'hyperbolicité complexe, qui étudie la répartition des courbes entières dans des variétés, et plus généralement des valeurs des applications méromorphes entre variétés complexes. Les apports de ce mémoire concernent principalement deux problèmes. Premièrement, on souhaite étendre des concepts et des résultats de la théorie de Nevanlinna pour des courbes entières au cadre d'applications méromorphes définies sur des variétés plus générales, dites paraboliques. Ces variétés sont munies d'une fonction d'exhaustion plurisousharmonique, solution de l'équation de Monge-Ampère complexe homogène, qui induit naturellement une pseudo-métrique hermitienne. Cette classe inclut notamment  $\mathbb{C}^m$  (la fonction d'exhaustion est simplement la norme euclidienne) et les revêtements holomorphes ramifiés de  $\mathbb{C}^m$ . Nous démontrons notamment des résultats de type «Second Théorème Principal» et «Inégalité tautologique» pour des applications holomorphes  $f : M \rightarrow X$  d'une variété parabolique  $M$  dans une variété projective lisse  $X$  de dimension  $\dim X \geq \dim M$ . Ce type de résultats donne des estimations de la proximité de l'image de  $f$  à un diviseur  $D \subset X$  donné.

La deuxième partie de ces travaux de thèse est consacrée à l'étude des propriétés d'hyperbolicité de complémentaires de diviseurs dans l'espace projectif complexe  $\mathbb{P}^n$ . On sait déjà, par un résultat de Green, que le complémentaire de  $2n + 1$  hyperplans génériques dans  $\mathbb{P}^n$  est hyperbolique. De plus, si dans un certain sens, le cotangent logarithmique associé à une paire logarithmique est «le plus ample possible», alors le complémentaire est hyperbolique. On formalise cette idée par les notions de cotangent logarithmique «presque ample» et «ample modulo le bord». Ces deux notions sont définies par la minimalité d'un certain lieu base augmenté. Grâce entre autres à des arguments de géométrie projective duale, nous obtenons une condition nécessaire et suffisante sur l'arrangement d'hyperplans pour que le fibré cotangent logarithmique soit presque ample. Nous en déduisons en particulier que cette propriété est satisfaite lorsque l'on considère au moins  $4n - 2$  hyperplans génériques. Nous obtenons le même type de résultats dans le cadre orbifolde et donnons quelques applications en hyperbolicité.

# Abstract

This thesis deals with the theory of complex hyperbolicity, which studies the geometry of entire curves in complex manifolds, and more generally value distribution of meromorphic mappings between complex manifolds. The contributions of this memoir concerns two main questions. First, we wish to extend concepts and results of Nevanlinna theory of value distribution for entire curves to the framework of parabolic manifolds. Those are manifolds endowed with a plurisubharmonic exhaustion function, solution of the homogeneous complex Monge-Ampère equation, which naturally induces a hermitian pseudo-metric. This class includes for instance  $\mathbb{C}^m$  and holomorphic ramified covers of  $\mathbb{C}^m$ . We prove a ‘Second Main Theorem’-type inequality and a Tautological inequality for holomorphic maps  $f : M \rightarrow X$  from a parabolic manifold  $M$  into a smooth projective variety  $X$  of dimension  $\dim X \geq \dim M$ . Such results provide estimations of the size of  $f(M) \cap D$  for a given divisor  $D$ .

The second part of this work is devoted to the study of complements of divisors in  $\mathbb{P}^n$  and their hyperbolicity properties. We already know, by a result of Green, that the complement of  $2n + 1$  generic hyperplanes in  $\mathbb{P}^n$  is hyperbolic. Moreover, when the logarithmic cotangent bundle has in some sense the strongest positivity property, then the complement is hyperbolic. We formalise this idea with the notions of ‘ample modulo boundary’ and ‘almost ample’ logarithmic cotangent bundle. These two notions are related to the minimality of some augmented base loci. We obtain a necessary and sufficient condition on the hyperplane arrangement so that the logarithmic cotangent bundle is almost ample. In particular, we deduce that the property holds when one has at least  $4n - 2$  generic components. We obtain similar results in the orbifold setting and give some applications to hyperbolicity.



*À mon grand-père Serge.*



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## **Bibliography**



# Introduction en français

## 1 La théorie de Nevanlinna : répartition des valeurs des applications holomorphes

### 1.1 Contexte historique

Tout mathématicien, quel que soit son domaine de recherche, se souvient d'avoir appris et utilisé le théorème de d'Alembert-Gauss, aussi appelé *théorème fondamental de l'algèbre*, dont une formulation moderne est la suivante : le corps  $\mathbb{C}$  des nombres complexes est algébriquement clos. En d'autres termes, considérant une fonction polynomiale  $f \in \mathbb{C}[z]$  et un nombre complexe  $a$ , l'équation  $f(z) = a$  admet exactement  $\deg f$  solutions (comptées avec multiplicités). La première mention du Théorème fondamental de l'algèbre sous cette forme, pour des polynômes réels, est attribuée à Peter Roth, en 1608. Il a été plus clairement énoncé par Descartes en 1637, en effectuant la distinction entre racines réelles et imaginaires. La première ébauche de preuve en a été donnée par D'Alembert en 1746, et le théorème fut intégralement démontré par Gauss, qui en publia plusieurs versions, utilisant des méthodes géométriques, en 1816 puis 1849.

Ce résultat s'étend aisément à une fonction rationnelle  $f = \frac{P}{Q} \in \mathbb{C}(z)$ , en définissant le degré de  $f$  par  $\deg f = \max\{\deg P, \deg Q\}$ . De la même façon, on peut montrer que  $f$  possède au plus  $2 \deg f - 1$  valeurs critiques. Plusieurs résultats permettent de localiser plus ou moins précisément les racines d'un polynôme, voire de les calculer explicitement : théorème de Landau-Mignotte, méthode de Descartes, suites de Sturm, mesure de Mahler...

Dès le XVII<sup>e</sup> siècle et les travaux de Descartes, le développement de la géométrie algébrique a naturellement suivi l'étude des polynômes en une variable. Une variété algébrique affine (resp. projective) est définie comme le lieu des zéros d'un ou plusieurs polynômes (resp. homogènes) en plusieurs variables. Les premiers travaux ont concerné les courbes algébriques dans le plan, définies par un polynôme  $P \in k[X, Y]$ , où  $k$  est un corps. On peut considérer la recherche des racines d'un polynôme  $P \in k[X]$  comme un problème de géométrie algébrique plane : cela revient à étudier l'intersection des deux courbes algébriques définies sur  $k$  par les équations  $Y = 0$  et  $Y = P(X)$ . Plus généralement, on souhaite décrire l'ensemble des points d'intersection de deux courbes algébriques, c'est-à-dire résoudre un système de deux équations polynomiales. Comme pour le nombre de racines d'un polynôme, on peut espérer que le nombre de ces points d'intersection ne dépende généralement que des degrés des deux courbes. Un cas particulier de ce problème fut étudié par Descartes dans le cadre du calcul de la droite tangente à une courbe en

un point, par le biais de la recherche du cercle osculateur. La méthode consiste à écrire l'équation des cercles passant par le point de la courbe et à chercher celui des cercles qui n'a qu'un point d'intersection unique avec la courbe. Par la suite, dès le XVIII<sup>e</sup> siècle, la recherche du nombre de points d'intersection de deux courbes algébriques planes se fait par élimination d'une des deux variables. Dans son ouvrage *Geometria Organica*, publié en 1720, Maclaurin conjectura, sans pouvoir le démontrer, que pour des courbes planes définies par des polynômes de degrés respectifs  $m$  et  $n$ , «en général, le nombre de points d'intersection est égal à  $mn$ ». La question est aussi étudiée dans certains cas particuliers par Euler. La première preuve de ce résultat, dans le cas de multiplicités simples, fut donnée par Bézout en 1764. Un des outils de la preuve originale est le résultant des deux polynômes, utilisé pour éliminer une inconnue. La formulation moderne du théorème est la suivante.

### **Théorème 0.1.1**

*Soient  $C, C'$  deux courbes algébriques projectives planes de degrés respectifs  $m, n$ , définies sur un corps algébriquement clos, sans composante commune. Alors le nombre de points d'intersection de  $C$  et  $C'$ , comptés avec multiplicité, est égal à  $mn$ .*

Dès lors, plusieurs questions se posent.

- *Quid* des dimensions supérieures, à savoir l'intersection d'hypersurfaces algébriques ?
- Que peut-on dire si les courbes ne sont plus algébriques ?

Les fonctions holomorphes ou analytiques, objets centraux de l'analyse complexe, peuvent être vues comme une généralisation des polynômes. De fait, les mathématiciens se sont naturellement intéressés aux zéros de ces fonctions. Un premier résultat est le principe des zéros isolés, qui affirme que les zéros d'une fonction holomorphe non identiquement nulle, définie sur un ouvert connexe de  $\mathbb{C}$ , forment un ensemble discret. Ainsi, pour  $r > 0$ , une fonction entière non nulle a un nombre fini de zéros à l'intérieur du disque  $D(0, r)$ , qui peut être calculé à l'aide du principe de l'argument ou du théorème de Rouché (conséquences de la formule de Cauchy).

De ce résultat découle aussi le principe du prolongement analytique, selon lequel une fonction holomorphe sur un ouvert  $U \subset \mathbb{C}$  est entièrement déterminée par ses valeurs sur n'importe quel sous-ensemble  $A$  de  $U$  possédant un point d'accumulation dans  $U$  (résultat analogue au fait qu'un polynôme de degré  $n$  est entièrement déterminé par ses valeurs en  $n + 1$  points). Cette théorie permet notamment de montrer que les fonctions  $L$  de Dirichlet, parmi lesquelles la fonction  $\zeta$  de Riemann, peuvent être prolongées en des fonctions holomorphes sur  $\mathbb{C} \setminus \{1\}$ .

Emile Picard fit progresser la connaissance de la répartition des valeurs des fonctions holomorphes avec deux résultats essentiels.

### **Théorème 0.1.2 (Petit Théorème de Picard)**

*Toute fonction entière non constante prend chaque valeur au moins une fois, avec au plus une exception.*

### **Théorème 0.1.3 (Grand Théorème de Picard)**

*Toute fonction holomorphe ayant une singularité essentielle prend chaque valeur une infinité de fois au voisinage de cette singularité, avec au plus une exception.*



## 1.2 En une variable

Poursuivant les travaux de Picard et Borel, Rolf Nevanlinna développa une théorie, portant désormais son nom, de la répartition des valeurs des fonctions méromorphes. Étant donné une fonction méromorphe  $f$  et un point  $a$  de la droite projective  $\mathbb{P}^1(\mathbb{C})$ , l'idée générale de la théorie est d'estimer asymptotiquement la taille de l'image réciproque  $f^{-1}(\{a\}) \cap \{|z| < r\}$  à l'aide notamment d'une fonction quantifiant la vitesse de croissance de la fonction  $f$ , à savoir la *fonction caractéristique de Nevanlinna*  $T(r, f)$ . Ces résultats d'estimation sont connus comme les trois *Théorèmes principaux de Nevanlinna*.

Soit  $E = \sum_i m_i(a_i)$  un diviseur dans  $\mathbb{C}$ . On définit les *fonctions de comptage*

$$n(E, t) = \sum_{|a_i| < t} m_i \quad t > 0$$

$$N(E, r) = \int_1^r n(E, t) \frac{dt}{t}, r > 1.$$

Pour tout  $a \in \mathbb{P}^1(\mathbb{C})$ , on souhaite donner une estimation asymptotique du nombre  $n((f - a)_0, r)$  de solutions (comptées avec multiplicité) de l'équation

$$f(z) = a$$

dans le disque ouvert  $\Delta_r = \Delta(0, r)$ . Ici,  $(f - a)_0$  est le diviseur des zéros de  $f - a$ .

Plus précisément, la fonction  $N_{f,a}(r) := N((f - a)_0, r)$  calcule le nombre moyen d'antécédents de  $a$  dans  $\Delta_r$ .

La *fonction d'approximation*

$$m_{f,a}(r) = \int_{|z|=r} \log^+ \left( \frac{1}{|f(z) - a|} \right) dz$$

décrit la proximité  $f$  à  $a$  sur le cercle  $\partial\Delta_r$ . On voit en effet que  $m_{f,a}(r)$  est une fonction décroissante de  $\|f - a\|_{L^1(\Delta_r)}$ .

Le *Premier théorème principal de Nevanlinna* affirme que la somme  $N_{f,a}(r) + m_{f,a}(r)$  est essentiellement indépendante du choix du point  $a$  lorsque  $r$  croît vers  $+\infty$  (à un terme borné près, lequel dépend de  $a$ ). De plus, cette somme est liée au volume de l'image de  $f$ . Plus précisément, définissons la fonction caractéristique

$$T(r, f) = N_{f,\infty} + m_{f,\infty}(r) = N_{1/f,0}(r) + m_{1/f,0}(r).$$

### Théorème 0.1.4 (Premier Théorème principal)

Pour tout  $a \in \mathbb{P}^1(\mathbb{C})$ ,

$$T\left(r, \frac{1}{f - a}\right) = T(r, f) + O(1),$$

où  $O(1)$  représente un terme borné (dépendant de  $a$  et  $f$ ).

Comme conséquence immédiate, on peut majorer la fonction de comptage par la fonction caractéristique.

**Proposition 0.1.5 (Inégalité de Nevanlinna)**

Il existe une constante  $C$  telle que pour tout  $a \in \mathbb{P}^1$ ,

$$N_{f,a}(r) \leq T(r, f) + C.$$

La fonction caractéristique peut s'interpréter géométriquement comme une mesure du volume de  $f(\mathbb{C})$  dans  $P^1(\mathbb{C})$ . Soit en effet  $\omega_{FS} = dd^c \log(1 + |w|^2)$  la métrique de Fubini-Study sur  $\mathbb{P}^1(\mathbb{C})$ . On introduit la fonction caractéristique d'Ahlfors-Shimizu

$$T_{f,\omega_{FS}}(r) = \int_1^r \frac{dt}{t} \int_{\Delta_r} f^* \omega_{FS} = \int_1^r \frac{dt}{t} \int_{\Delta_r} dd^c \log(1 + |f(z)|^2).$$

Cette fonction quantifie la taille moyenne de l'image  $f(\Delta_t)$  du disque de rayon  $t$ . Le calcul montre que

$$T_{f,\omega_{FS}}(r) = T(r, f) + O(1).$$

De plus, on obtient la formule de Crofton

$$T_{f,\omega_{FS}}(r) = \int_{w \in \mathbb{P}^1(\mathbb{C})} N_{f,w}(r) \omega_{FS}.$$

Il se trouve que la fonction  $T(r, f)$  peut être vue comme une généralisation du degré d'une fonction rationnelle au cas transcendant, de la façon suivante.

**Définition 0.1.6**

L'ordre  $\rho(f)$  d'une fonction entière  $f$  est l'entier

$$\limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}.$$

On calcule par exemple

- si  $f = P/Q$  est une fonction rationnelle ( $P, Q$  premiers entre eux),

$$T(r, f) = \max\{\deg P, \deg Q\} \log r + O(1) \text{ et } \rho(f) = 0;$$

- pour tout  $p \in \mathbb{N}$ ,  $T(r, e^{z^p}) = \frac{r^p}{\pi}$  et  $\rho(e^{z^p}) = p$ .

On tire d'ailleurs de la fonction caractéristique un critère de rationalité pour les fonctions méromorphes.

**Proposition 0.1.7**

Une fonction méromorphe  $f$  est rationnelle si et seulement si  $T(r, f) = O(\log r)$ .

Pour donner une autre application de l'ordre  $\rho$ , rappelons que le théorème fondamental de l'algèbre permet d'écrire tout polynôme complexe comme un produit de facteurs de degré 1. Il existe aussi un théorème de décomposition, dû entre autres à Hadamard, pour les fonctions méromorphes, qui généralise le cas polynomial.

**Théorème 0.1.8 (Hadamard)**

Soient  $f$  une fonction méromorphe d'ordre fini  $\rho$ ,  $(a_n)$  et  $(b_n)$  les suites respectivement de ses zéros et pôles non nuls (deux à deux distincts). Définissons les facteurs primaires de Weierstrass comme les fonctions entières  $E(z, m) = (1 - z) \exp(z + z^2/2 + \dots + z^m/m)$ . Alors il existe deux entiers  $m_1$  et  $m_2$  inférieurs à  $\rho$  et un polynôme  $h$  de degré  $\leq \rho$  tels que

$$f(z) = e^{h(z)} z^m \frac{\prod_n E(z/a_n, m_1)}{\prod_n E(z/b_n, m_2)}$$

où  $m$  désigne l'ordre d'annulation de  $f$  en 0.

On peut interpréter le Premier théorème comme le fait que la « proximité » totale de  $f(\mathbb{C})$  à un point  $a \in \mathbb{P}^1$  fixé est essentiellement indépendante du choix de  $a$ . Le Second Théorème Principal de Nevanlinna est un résultat opposé en un sens, en ce qu'il donne une majoration de la fonction caractéristique par des fonctions de comptage.

On introduit une notation spécifique : si  $\phi$  et  $\psi$  sont deux fonctions positives définies sur un intervalle  $[r_0, +\infty[$ ,  $r_0 \geq 0$ , on écrit

$$\phi(r) \leq_{exc} \psi(r)$$

si l'inégalité est satisfaite en-dehors d'un borélien de mesure de Lebesgue finie.

**Proposition 0.1.9 (Lemme de la dérivée logarithmique)**

Soit  $f$  une fonction méromorphe sur  $\mathbb{C}$ . Alors pour tout  $\delta > 0$ ,

$$m(r, f'/f) \leq_{exc} \left(1 + \frac{(1 + \delta)^2}{2}\right) \log^+ T(r, f) + S(r, f)$$

où  $S(r, f) = O(\log T(r, f) + \log r)$ .

Le lemme de la dérivée logarithmique implique le Second Théorème Fondamental de la théorie de Nevanlinna, qui est en quelque sorte une inégalité opposée à l'inégalité de Nevanlinna.

**Théorème 0.1.10 (Second Théorème Principal)**

Soient  $f$  une fonction méromorphe sur  $\mathbb{C}$  et  $a_1, \dots, a_q$  des points distincts de  $\mathbb{P}^1$ . Alors

$$(q - 2)T(r, f) \leq_{exc} \sum_{i=1}^q N_{f, a_i}^{[1]}(r) + S(r, f).$$

Ici, si  $E$  est un diviseur effectif dans  $\mathbb{C}$ , la fonction de comptage tronquée est

$$N_E^{[1]}(r) = N_{E_{red}}(r)$$

Ainsi, on peut contrôler le volume de  $f(\mathbb{C})$  grâce à la préimage de trois points quelconques. La *déficiencia* d'une fonction méromorphe  $f$  en un point  $a \in \mathbb{P}^1$  est définie par

$$\delta(f, a) = 1 - \limsup_{r \rightarrow \infty} \frac{N_{f, a}(r)}{T(r, f)}$$

$$\delta_1(f, a) = 1 - \limsup_{r \rightarrow \infty} \frac{N_{f, a}^{[1]}(r)}{T(r, f)}$$

Ces nombres vérifient

$$0 \leq \delta(f, a) \leq \delta_1(f, a) \leq 1.$$

Un point  $a$  est dit *valeur déficiente pour  $f$*  si  $\delta(f, a) > 0$ . En particulier, si  $f$  évite un point  $a$ ,  $\delta(f, a) = 1$ .

**Théorème 0.1.11 (Relation de déficience, Troisième Théorème principal)**

*Soit  $f$  une fonction méromorphe non constante. L'ensemble des valeurs déficientes pour  $f$  est au plus dénombrable et*

$$\sum_{a \in \mathbb{P}^1} \delta(f, a) \leq 2.$$

On retrouve ainsi le petit théorème de Picard.

Citons une variante de ce dernier théorème. Pour  $a \in \mathbb{P}^1$ , soit

$$\nu_a = \min\{\text{ord}_z(f - a); z \in f^{-1}(a)\}$$

la multiplicité minimale d'un antécédent de  $a$  par  $f$  (par convention,  $\nu_a = \infty$  si  $f$  évite  $a$ ).

**Proposition 0.1.12**

*Soit  $f$  une fonction méromorphe non constante. Alors*

$$\sum \left(1 - \frac{1}{\nu_a}\right) \leq 2.$$

*Par conséquent, il y a au plus quatre valeurs de  $a$  pour lesquelles  $\nu_a \geq 2$ .*

### 1.3 Courbes entières dans une variété projective

Une théorie de Nevanlinna en plusieurs variables s'est alors naturellement développée sous l'impulsion de travaux de Bloch, Cartan, H. Weyl-J.Weyl, et l'étude des courbes entières dans des variétés quasi-projectives est encore un champ de recherches très actif de nos jours. Déjà évoquée précédemment, la théorie algébrique de l'intersection est un des sujets centraux de la géométrie algébrique. D'une certaine façon, on peut interpréter la théorie de Nevanlinna comme une version transcendante de la théorie de l'intersection : on souhaite décrire l'intersection de diviseurs avec des courbes entières, plus nécessairement algébriques. Dans le cadre transcendant, plusieurs objets vont remplacer le nombre d'intersection. La fonction de comptage et la fonction de proximité décrivent la position d'une courbe entière par rapport à un diviseur. La fonction caractéristique de Nevanlinna quantifie d'une certaine manière le volume de la courbe entière. L'enjeu de la théorie de Nevanlinna est de comparer ces trois fonctions. Un des principaux objectifs est ainsi d'obtenir des résultats de type « Second Théorème Principal » tel que présenté ci-dessous.

**Conjecture 0.1.13 ( [88], Conj. 4.10.2)** *Soient  $X$  une variété projective lisse,  $D$  un diviseur à croisements normaux et  $A$  un fibré en droites ample sur  $X$ . Alors il existe  $k_0 \in \mathbb{N}$  tel que pour toute courbe entière  $f : \mathbb{C} \rightarrow X$  Zariski-dense, et pour tout  $\varepsilon > 0$  suffisamment petit, l'inégalité suivante soit vérifiée en-dehors d'un borélien de mesure de Lebesgue finie.*

$$T_{f, \mathcal{O}_X(D)}(r) + T_{f, K_X}(r) \leq N_{f, D}^{[k_0]}(r) + \varepsilon T_{f, A}(r).$$

Auquel cas, on dira que la paire  $(X, D)$  vérifie le Second théorème (SMT) tronqué à l'ordre  $k_0$ .

Une version plus faible, et cependant toujours conjecturale, avait été énoncée par Griffiths.

**Conjecture 0.1.14 (Griffiths)** *Dans le même contexte que 0.1.13,*

$$T_{f, \mathcal{O}_X(D)}(r) + T_{f, K_X}(r) \leq N_{f, D}(r) + \varepsilon T_{f, A}(r).$$

Des résultats ont été obtenus dans des cas particuliers. Citons par exemple

- théorème de Cartan : hyperplans en position générale dans  $\mathbb{P}^n$
- Eremenko-Sodin : SMT sans troncatures dans  $\mathbb{P}^n$
- Ru : SMT sans troncatures pour des courbes entières non algébriquement dégénérées,  $X$  projective
- Steinmetz, Ru-Stoll, Yamanoi : SMT pour une famille de diviseurs (*moving targets*)
- McQuillan, Noguchi-Winkelmann-Yamanoi : courbes entières dans une variété semi-abélienne

Par la suite, des pistes de généralisation furent proposées :

- Shiffmann : cas d'un diviseur  $D$  plus singulier
- Campana, Rousseau ( [99], [19] ) : paires orbifoldes.
- A-t-on une théorie similaire si l'on remplace les courbes entières dans une variété algébrique  $X$  par des applications holomorphes  $f : M \rightarrow X$  ? Si, dans l'immense majorité des cas, cette dernière question reste en suspens, des résultats ont été obtenus lorsque  $M = \mathbb{C}^m$  (plus ou moins forts selon la valeur de  $m$ ), et plus généralement pour des variétés paraboliques non compactes, qui sont des variétés munies d'une fonction d'exhaustion annulant un opérateur de Monge-Ampère et définissant une pseudo-métrique sur  $M$ . Cette direction fut notamment explorée par Ahlfors [1], Stoll [120], Wong, Kobayashi, Carlson-Griffiths, Griffiths-King, Paun-Sibony [93], Brotbek-Brunebarbe [12].

Outre l'intérêt intrinsèque de ces questions, une des raisons pour lesquelles l'étude des courbes entières est encore vivace est leur lien avec la théorie algébrique des nombres. En effet, P. Vojta remarqua une analogie avec la théorie de l'approximation diophantienne, en l'interprétant comme l'étude des courbes holomorphes sur des variétés définies sur des corps de fonctions adéquats. De ce point de vue, la théorie de Nevanlinna classique peut être vue comme un problème d'approximation des nombres complexes par des fonctions méromorphes transcendentes. On obtient alors un dictionnaire complet entre les objets de la théorie de Nevanlinna et ceux de l'approximation diophantienne. Dans ce cadre, la conjecture *abc* serait la conséquence d'un analogue du Second théorème. Pour de plus amples détails, on se réfère à l'ouvrage [128].

## 2 Hyperbolicité

### Différentes notions d'hyperbolicité

Comme d'une certaine façon la théorie de l'intersection dans le cas algébrique, la théorie de Nevanlinna cherche à décrire de manière quantitative la position d'une courbe entière dans une variété, au moyen des fonctions de comptage et de répartition. En parallèle, d'autres auteurs se sont intéressés aux courbes entières d'un point de vue plus géométrique : c'est la théorie de l'hyperbolicité complexe. En géométrie complexe, le terme *hyperbolicité* désigne plusieurs propriétés des variétés complexes qui, comme on va le voir, sont fortement liées entre elles.

De façon générale, on peut obtenir beaucoup d'informations sur la géométrie d'une variété  $X$  par le biais des courbes qu'elle contient. L'étude des courbes rationnelles joue un rôle crucial dans la classification birationnelle des variétés algébriques. Lorsque la variété  $X$  n'est plus nécessairement algébrique, les objets considérés seront les courbes entières. Plusieurs questions émergent alors naturellement.

1. Y a-t-il des courbes entières (non constantes) à valeurs dans  $X$  ?
2. Existe-t-il une sous-variété propre  $Z \subset X$  contenant toutes les courbes entières  $f : \mathbb{C} \rightarrow X$  ?
3. Si  $f$  est une courbe entière dans  $X$ , son image  $f(\mathbb{C})$  est-elle Zariski-dense ?

La première définition de l'hyperbolicité est directement liée aux courbes entières.

#### Définition 0.2.1

*Une variété complexe  $X$  est dite hyperbolique au sens de Brody si elle ne contient aucune courbe entière (non constante). Plus généralement,  $X$  est dite (Brody-)hyperbolique modulo un sous-ensemble algébrique  $Z$  si toute courbe entière  $f : \mathbb{C} \rightarrow X$  a son image contenue dans  $Z$ .*

Par exemple, on peut reformuler le théorème de Picard en disant que  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$  est Brody-hyperbolique.

La deuxième notion principale d'hyperbolicité est celle qui explique le plus le choix du terme *hyperbolicité*, qui qualifie en géométrie riemannienne une métrique de courbure négative. Cette notion fut introduite par S. Kobayashi.

#### Définition 0.2.2

*Soit  $X$  une variété complexe lisse. On note  $d_P$  la distance de Poincaré sur le disque unité  $\Delta$ .*

*On définit la pseudo-distance de Kobayashi  $d_X$  comme la plus petite pseudo-distance  $d$  sur  $X$  rendant décroissantes toutes les applications holomorphes  $f : \Delta \rightarrow X$  :*

$$\forall x, y \in \Delta, d(f(x), f(y)) \leq d_P(x, y).$$

Naturellement,  $d_\Delta$  coïncide avec la distance de Poincaré (conséquence du lemme de Schwarz). En revanche,  $d_{\mathbb{C}} \equiv 0$ .

**Définition 0.2.3**

Une variété  $X$  est dite Kobayashi-hyperbolique si  $d_X$  est une distance, c'est-à-dire

$$\forall x, y \in X, x \neq y \implies d_X(x, y) > 0.$$

On peut montrer [70] que si une variété complexe  $X$  possède une métrique de courbure sectionnelle holomorphe majorée par une constante strictement négative, alors  $X$  est Brody-hyperbolique (d'où l'utilisation du terme *hyperbolique*). En particulier, en dimension 1, les surfaces de Riemann hyperboliques sont exactement celles dont le fibré tangent peut être muni d'une métrique hermitienne de courbure négative (ce sont les surfaces de Riemann uniformisées par le disque unité, et la métrique en question est celle induite par la métrique de Poincaré).

On voit aisément que l'hyperbolicité au sens de Kobayashi implique l'hyperbolicité au sens de Brody. Le lemme de reparamétrisation de Brody affirme que les deux notions sont en réalité équivalentes pour une variété *compacte*.

**Hyperbolicité et variétés de type général****Hyperbolicité des hypersurfaces de grand degré**

Le théorème de Green [58] généralise le petit théorème de Picard en dimension supérieure.

**Théorème 0.2.4**

*Il n'existe pas d'application holomorphe non constante*

$$f : \mathbb{C} \rightarrow \mathbb{P}^n \setminus H_1 \cup \dots \cup H_{2n+1}$$

*à valeurs dans le complémentaire de  $2n + 1$  hyperplans en position générale.*

Green prouva en fait un résultat plus fort de dégénérescence algébrique.

**Théorème 0.2.5**

*L'image d'une courbe entière  $f : \mathbb{C} \rightarrow \mathbb{P}^n \setminus H_1 \cup \dots \cup H_{n+k}$  évitant  $n + k \geq n + 1$  hyperplans en position générale est contenue dans un sous-espace linéaire de  $\mathbb{P}^n$  de dimension au plus  $\left\lfloor \frac{n}{k} \right\rfloor$ .*

Une fois que le problème des hyperplans fut résolu, il était naturel de chercher d'autres exemples de variétés hyperboliques comme complémentaires d'hypersurfaces plus générales. La conjecture de Kobayashi prédit que la borne de Green reste valable pour des hypersurfaces autres que des unions d'hyperplans.

**Conjecture 0.2.6 (Kobayashi)** *Le complémentaire d'une hypersurface générale  $X_d$  de degré  $d \geq 2n + 1$  dans  $\mathbb{P}^n$  est Brody-hyperbolique.*

## Conjectures de Green-Griffiths-Lang

Le programme du modèle minimal exprime le lien entre la géométrie d'une variété complexe et son fibré canonique  $K_X$ . Rappelons qu'une variété compacte est *de type général* si son fibré canonique est *gros*, c'est-à-dire si la dimension des espaces de sections pluri-canoniques a un taux de croissance maximal :

$$h^0(X, K_X^{\otimes m}) \geq Cm^n$$

pour une certaine constante  $C > 0$ . On sait relativement peu de choses sur les variétés de type général. La conjecture de Green-Griffiths-Lang prédit que les variétés de type général devraient avoir de fortes propriétés d'hyperbolicité.

**Conjecture 0.2.7 (Green-Griffiths-Lang)** *Soit  $X$  une variété projective lisse de type général. Alors il existe une sous-variété propre  $Z \subsetneq X$  contenant toutes les courbes entières  $f : \mathbb{C} \rightarrow X$ .*

En conséquence, on pourrait caractériser les variétés hyperboliques comme celles dont toutes les sous-variétés sont de type général.

Remarquons au passage que cette conjecture concerne en particulier les courbes rationnelles et elliptiques dans  $X$ . Dans le cas d'une surface de type général, la sous-variété  $Z$  serait de dimension au plus 1. De fait,  $X$  ne contiendrait qu'un nombre fini de courbes rationnelles ou elliptiques. Même cette conséquence plus faible reste ouverte.

La conjecture de Green-Griffiths-Lang 0.2.7 impliquerait une autre conjecture, formulée par Lang.

**Conjecture 0.2.8 ([73])** *Soit  $X$  une variété projective lisse de type général. Alors il existe un sous-ensemble algébrique propre  $Z \subset X$  contenant toutes les sous-variétés propres (non nécessairement lisses) de  $X$  qui ne sont pas de type général.*

Dans cette thèse, on s'intéressera aussi à la version logarithmique de la conjecture de Green-Griffiths-Lang (pour des variétés quasi-projectives).

**Conjecture 0.2.9** *Soit  $X$  une variété projective lisse et  $D \subset X$  un diviseur à croisements normaux simples. Alors les deux assertions ci-dessous sont équivalentes.*

- *La paire logarithmique  $(X, D)$  est de type log-général :  $K_X + D$  est un diviseur gros.*
- *Il existe une sous-variété propre  $Z \subset X$  contenant toutes les courbes entières  $f : \mathbb{C} \rightarrow X \setminus D$  telle que  $f(\mathbb{C}) \not\subset Z$ .*

Le lien avec la théorie de Nevanlinna est assez clair : ce sont deux approches différentes des courbes entières, mais néanmoins liées. En effet, supposons que le Second Théorème Principal tronqué à l'ordre 1 soit vérifié par une paire logarithmique lisse  $(X, D)$  de type log-général. S'il existe une courbe entière  $f : \mathbb{C} \rightarrow X \setminus D$ , la fonction de comptage  $N_{f,D}(r)$  est identiquement nulle. Comme  $K_X(D)$  est gros, la fonction caractéristique associée  $T_{f,K_X(D)}(r)$  est positive et croissante en  $r$ . Mais cela est rendu impossible par le Second théorème.



### Quelques résultats

Ces conjectures restent aujourd'hui largement ouvertes. Depuis les travaux de Bloch sur les courbes entières dans des variétés abéliennes, la principale approche pour traiter ces problèmes consiste à produire des équations différentielles algébriques qui devront être satisfaites par les courbes entières. Si on a suffisamment d'équation indépendantes, les contraintes sur les courbes entières seront alors trop nombreuses. On formalise cette idée par la notion de jets. Si  $f = [f_0 : \cdots : f_n] : \mathbb{C} \rightarrow X$  est une application holomorphe, son  $k$ -jet en un point  $x$  représente son développement de Taylor en 0 dans une carte locale  $(X, x) \cong (\mathbb{C}^n, 0)$ . Les *différentielles de jets* d'ordre  $k$  et de poids  $m$  sont obtenues en recollant des polynômes homogènes de poids homogène  $m$  (pour l'action de  $\mathbb{C}^*$ ) en les coefficients de Taylor  $f_i^{(j)}$ ,  $0 \leq i \leq n, 1 \leq j \leq k$  de sorte à former un fibré vectoriel  $E_{k,m}^{GG}\Omega_X$ .

Les différentielles de jets vérifient le théorème fondamental d'annulation suivant.

#### **Théorème 0.2.10**

Soient  $X$  une variété projective lisse,  $\mathbb{P} \in H^0(X, E_{k,m}^{GG}\Omega_X \otimes A^{-1})$  une différentielle de jets holomorphe s'annulant le long d'un diviseur ample  $A \subset X$ , et  $f : \mathbb{C} \rightarrow X$  une courbe entière. Alors

$$f^*P \equiv 0$$

sur  $\mathbb{C}$ .

Il existe aussi une version logarithmique de ce résultat.

#### **Théorème 0.2.11**

Soient  $(X, D)$  une paire logarithmique lisse,  $\mathbb{P} \in H^0(X, E_{k,m}^{GG}\Omega_X(\log D) \otimes A^{-1})$  une différentielle de jets logarithmique s'annulant le long d'un diviseur ample  $A \subset X$ , et  $f : \mathbb{C} \rightarrow X \setminus D$  une courbe entière. Alors

$$f^*P \equiv 0$$

sur  $\mathbb{C}$ .

Autrement dit, chaque section globale d'un fibré  $E_{k,m}^{GG}\Omega_X \otimes A^{-1}$  (resp.  $E_{k,m}^{GG}\Omega_X(\log D) \otimes A^{-1}$  dans le cas logarithmique) imposera une condition supplémentaire sur les courbes entières dans  $X$  (resp.  $X \setminus D$ ).

Par la suite, Siu proposa une technique générale (dérivée d'idées de Clemens, Ein et Voisin) pour produire un grand nombre de différentielles de jets holomorphes dans le cadre des hypersurfaces, en déformant des sections explicites. La méthode de Siu, combinée à d'autres comme des arguments cohomologiques notamment dus à Demailly et Diverio, fut par exemple utilisée dans des travaux de Paun en dimension 2 ([92]), Rousseau ([96], [97], [98]), Diverio-Merker-Rousseau ([44]).

## 3 Contenu de la thèse

Nous résumons maintenant le contenu de la thèse elle-même en donnant les principaux résultats et idées. Nous travaillons exclusivement sur le corps  $\mathbb{C}$  des nombres complexes. Sauf indication contraire,  $X$  désignera une variété complexe projective lisse de dimension  $n \geq 1$ .

## Chapitre 1

Le premier chapitre rassemble des définitions et résultats préliminaires dont nous aurons besoin tout au long de cette thèse.

Nous commençons par évoquer quelques aspects de géométrie différentielle, au sujet des pseudo-métriques hermitiennes définies sur les variétés complexes. Soit  $h$  une pseudo-métrique hermitienne lisse sur  $X$ , et soit  $\omega$  la  $(1, 1)$ -forme semi-positive associée. On définit sa *courbure de Ricci* comme la  $(1, 1)$ -forme

$$\text{Ric } \omega = -2\pi dd^c \log \lambda = -i \sum_{i,j} \frac{\partial^2 \log \lambda}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j,$$

où  $\lambda \in \mathcal{C}^\infty(X, \mathbb{R})$  est la fonction satisfaisant

$$\omega^n = \left(\frac{i}{2}\right)^n \lambda \prod_{1 \leq j \leq n} (dz_j \wedge d\bar{z}_j).$$

*A priori*, cette expression a un sens uniquement sur le complémentaire du lieu de dégénérescence de  $h$ . Sous certaines hypothèses sur les singularités de la métrique  $h$ , on peut en fait étendre  $\text{Ric } \omega$  en un courant sur la variété  $X$  toute entière. Nous donnons ensuite quelques résultats de base sur les fonctions pluri-sous-harmoniques.

Dans une deuxième section, nous rappelons les notions essentielles de positivité pour les fibrés en droites holomorphes que nous considérerons par la suite. Nous rencontrerons principalement les notions de fibrés amples, gros et nef.

### Définition 0.3.1

Soit  $L$  un fibré en droites holomorphe sur  $X$ . Le fibré  $L$  est dit :

- très ample s'il existe un plongement  $\iota : X \hookrightarrow \mathbb{P}^N$  tel que  $L = \iota^* \mathcal{O}_{\mathbb{P}^N}(1)$  ;
- ample si une puissance tensorielle  $L^{\otimes m}$  est très ample ;
- gros s'il existe un entier  $m \geq 1$  et une application  $\iota : X \dashrightarrow \mathbb{P}^N$  birationnelle sur son image telle que  $L^{\otimes m} = \iota^* \mathcal{O}_{\mathbb{P}^N}(1)$ .

On définit les notions correspondantes pour les diviseurs sur  $X$ . Dans le groupe de Néron-Severi  $N^1(X)$ , les classes respectives des diviseurs amples et gros forment toutes deux des cônes ouverts  $\text{Amp}(X)$  et  $\text{Big}(X)$ . On pose  $\text{Nef}(X) = \overline{\text{Amp}(X)}$  et  $\text{Pseff}(X) = \overline{\text{Big}(X)}$ . Un diviseur est dit *nef* (resp. *pseudoeffectif*) si sa classe appartient à  $\text{Nef}(X)$  (resp.  $\text{Pseff}(X)$ ). Nous rappelons dans la suite de la section diverses caractérisations de ces propriétés et leurs manifestations cohomologique, métrique ou numérique.

Dans la section 1.3, nous expliquons comment étendre ces concepts à un fibré vectoriel  $E$  de rang supérieur en se ramenant au cas des fibrés en droites. À cet effet, on considère le projectivisé

$$\mathbb{P}(E) \xrightarrow{\pi_E} X$$

des quotients de rang 1 de  $E$ . On dit alors que  $E$  est ample / nef / gros si le fibré tautologique  $\mathcal{O}_{\mathbb{P}(E)}(1)$  possède la même propriété.

Soient  $L$  un fibré en droites sur  $X$  et  $A$  un fibré en droites ample. Le *lieu base augmenté* de  $L$  est le sous-ensemble fermé

$$\mathbb{B}_+(L) = \bigcap_{N,q \geq 1} \text{Bs}(L^{Nq} \otimes A^{-q}).$$

On montre aisément que  $L$  est gros ssi  $\mathbb{B}_+(L) \neq X$ , et ample ssi  $\mathbb{B}_+(L)$  est vide. De plus,  $\mathbb{B}_+(L)$  est bien indépendant du choix de  $A$ . Nous donnons plusieurs résultats sur les lieux base des fibrés. Une propriété du lieu base augmenté, corollaire d'un résultat (du à Nakamaye) relatif aux fibrés nef et gros sera notamment pour nous de première importance dans le chapitre 4.

### Proposition 0.3.2

*Soient  $Y$  une variété projective lisse et  $f : X \rightarrow Y$  un morphisme. Pour tout fibré en droites ample  $A$  sur  $Y$ , le lieu base augmenté de  $f^*A$  est la réunion des fibres de dimension strictement positive de  $f$  :*

$$\mathbb{B}_+(f^*A) = \{x \in X; \dim_x f^{-1}(\{f(x)\}) > 0\}.$$

Enfin, nous donnons une introduction à la théorie de l'hyperbolicité complexe et donnons quelques résultats reliant les différentes notions d'hyperbolicité.

## Chapitre 2

Le second chapitre est dédié à l'aspect *Théorie de Nevanlinna* de cette thèse. Il s'ouvre sur un résumé succinct des travaux originels de Nevanlinna [82] concernant la répartition des valeurs des fonctions méromorphes.

### Courbes entières dans les variétés complexes

La deuxième section se consacre à la théorie de Nevanlinna pour les courbes entières dans les variétés complexes. Nous y introduisons les objets qui généralisent naturellement les fonctions de Nevanlinna définies pour des fonctions méromorphes.

On fixe une variété projective  $X$ , un fibré en droites hermitien  $(L, h)$  sur  $X$ , un diviseur  $D \subset X$  à croisements normaux simples et  $f : \mathbb{C} \rightarrow X$  une application holomorphe telle que  $f(\mathbb{C}) \not\subset D$ . Écrivons  $f^*D = \sum_{i \in I} n_i(a_i)$ , pour  $a_i \in \mathbb{C}$ . La *fonction caractéristique de Nevanlinna associée à  $L$*  est

$$T_{f,L}(r) = \int_1^r \frac{dt}{t} \int_{\Delta_t} f^* c_1(L, h),$$

où  $\Delta_t$  est le disque de centre 0 et de rayon  $t$ . Cette définition est indépendante du choix de la métrique  $h$  à un terme borné près (qui sera sans importance pour les questions étudiées). Cette fonction représente le volume moyen de l'image  $f(\Delta_t)$  calculé par rapport à la  $(1,1)$ -forme  $c_1(L, h)$ .

La *fonction de comptage tronquée à l'ordre  $k$*

$$N_{f,D}^{[k]}(r) = \int_1^r \frac{dt}{t} \sum_{a_i \in \Delta_t} \max\{n_i, k\}$$

quantifie la taille moyenne de l'image  $f(\Delta_t) \cap D$ .

Comme évoqué plus haut, l'enjeu majeur de la théorie de Nevanlinna est d'établir des inégalités de type *Second théorème principal*, contraignant la position des courbes entières dans la variété  $X$ .

Dans la suite de cette section, nous citons des exemples de Seconds théorèmes déjà obtenus, et nous décrivons quelques applications de ce type de résultats.

### **Théorie de Nevanlinna pour les variétés paraboliques**

Dans la troisième section, on élargit les considérations de la théorie de Nevanlinna dans le cadre des variétés complexes paraboliques. Une variété parabolique est une variété complexe  $M$  munie d'une fonction d'exhaustion  $\sigma : M \rightarrow \mathbb{R}_+$  de classe  $\mathcal{C}^\infty$ , telle que  $\log \sigma$  soit plurisousharmonique et solution de l'équation de Monge-Ampère complexe homogène. La  $(1, 1)$ -forme  $dd^c \sigma$  définit alors une pseudo-métrique hermitienne sur  $M \setminus \sigma^{-1}(\{0\})$ . Cette classe de variétés inclut notamment les variétés affines, les revêtements holomorphes finis de  $\mathbb{C}^n$ , et les surfaces de Riemann non compactes n'admettant pas de fonction sous-harmonique majorée non constante. Suivant la théorie notamment développée par Stoll [121], nous montrons que l'on peut définir les objets de la théorie de Nevanlinna dans le cas parabolique.

Nous généralisons alors quelques résultats obtenus notamment par Paŭn-Sibony [93] et Brotbek-Brunebarbe [12] en dimension 1 et par Gasbarri-Pacienza-Rousseau [54] dans le cas  $\mathbb{C}^n$ . Nous obtenons notamment une *Inégalité tautologique* pour une variété parabolique  $M$ .

#### **Théorème 0.3.3**

Soit  $X$  une variété projective lisse de dimension  $n$ ,  $D \subset X$  un diviseur à croisements normaux simples et  $f : (M, \sigma) \rightarrow X$  une application holomorphe d'une variété parabolique  $(M, \sigma)$  de dimension  $m \leq n$  vers  $X$ , de rang maximal  $m$ . Soit

$$f_1 : M \rightarrow X_1 := \mathbb{P} \left( \bigwedge^m \Omega_X(\log D) \right)$$

le relèvement méromorphe naturel de  $f$ . Alors

$$T_{f_1, \mathcal{O}_{X_1}(1)}(r) \leq_{\text{exc}} N_{f, D}^{[1]}(r) - \text{Ric}_\sigma(r) + O(\log T_{f, \omega}(r)).$$

De plus, nous montrons un Second théorème dans le cas où la variété cible  $X$  est munie d'une métrique de courbure sectionnelle holomorphe strictement négative.

#### **Théorème 0.3.4**

Soit  $X$  une variété projective lisse de dimension  $n$  et  $D \subset X$  un diviseur. On munit  $X \setminus D$  d'une pseudo-métrique hermitienne lisse  $h$  dont la courbure sectionnelle holomorphe est majorée par une constante  $-\gamma < 0$  et dont la courbure bisectionnelle holomorphe est négative. On suppose que le lieu de dégénérescence  $\Sigma_h$  de  $h$  est analytique et de mesure nulle dans  $X \setminus D$ .

Soit  $(M, \sigma)$  une variété parabolique de dimension  $m \leq n$  et  $f : M \rightarrow X$  une application holomorphe telle que  $f(M) \not\subset D$  et  $f(M) \not\subset \Sigma_h$ . Notons  $\omega$  la  $(1, 1)$ -forme associée à la pseudo-métrique  $f^*h$  sur  $M \setminus f^{-1}(D)_{red}$ . Alors  $\omega$  est localement intégrable sur  $M$  et

$$\gamma T_{[\omega]}(r) \leq_{exc} 2\pi N_{f,D}^{[1]}(r) - \text{Ric}_\sigma(r) + O(\log T_{[\omega]}(r) + \log r).$$

## Chapitre 3

Le troisième chapitre est dévolu à des rappels sur la théorie des orbifolds géométriques de Campana. Nous introduisons la notion de paire orbifolde, qui représente en quelque sorte une interpolation entre le cas d'une variété compacte et celui d'une paire logarithmique. À savoir, une paire orbifolde lisse  $(X, \Delta)$  est la donnée d'une variété projective lisse  $X$  et d'un  $\mathbb{Q}$ -diviseur  $\Delta$  à coefficients dans  $[0, 1]$  dont le support est à croisements normaux simples.

Nous verrons comment ces objets permettent de modéliser un revêtement ramifié d'une variété compacte au-dessus d'un diviseur à croisements normaux.

Nous expliquerons ensuite comment construire des faisceaux de formes différentielles orbifoldes sur  $(X, \Delta)$  qui soient la généralisation idoine des formes à pôles logarithmiques. Soit  $(U, z_1, \dots, z_n)$  un système local de coordonnées tel que

$$\Delta \cap U = (z_1^{a_1} \cdots z_n^{a_n} = 0),$$

où les  $a_i$  sont des rationnels de  $[0, 1]$ . Formellement, on souhaite définir un faisceau de 1-formes à pôles fractionnaires, engendré par les expressions  $dz_i/z_i^{a_i}$ . Comme nous le décrirons, on peut donner un sens à ces objets par l'intermédiaire de revêtements ramifiés adaptés.

De nombreux attributs et propriétés des paires logarithmiques peuvent être généralisés au cas orbifolde. On s'intéresse naturellement aux courbes orbifoldes, qui sont les morphismes  $f : \mathbb{C} \rightarrow X$  ramifiant avec la multiplicité adéquate au-dessus du  $\Delta$ . Dès lors, un certain nombre de concepts et de questions d'hyperbolicité ont leurs équivalents dans le monde orbifolde.

## Chapitre 4

Dans le quatrième et dernier chapitre, nous nous intéressons à l'hyperbolicité du complémentaire d'un arrangement d'hyperplans  $D = H_1 + \cdots + H_c$  dans  $\mathbb{P}^n$ . Rappelons qu'à la suite de plusieurs travaux, il est désormais connu que le complémentaire de  $c \geq 2n + 1$  hyperplans en position générale dans  $\mathbb{P}^n$  est hyperbolique et hyperboliquement plongé dans  $\mathbb{P}^n$ .

De plus, le théorème fondamental d'annulation indique que l'on peut contrôler la répartition des courbes entières à l'aide du lieu base augmenté du fibré cotangent logarithmique

$$p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}^n(\log D)}(1))) \subset \mathbb{P}^n.$$

Aussi, bien que l'on sache déjà sous quelle condition le complémentaire  $\mathbb{P}^n \setminus D$  est hyperbolique, il est naturel de chercher à comprendre plus en profondeur ce lieu base augmenté.

D'emblée, on constate que le cotangent logarithmique ne peut être ample dès lors que  $D \neq \emptyset$ , et par conséquent

$$\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)) \neq \emptyset.$$

Plus précisément, le fibré  $\Omega_{\mathbb{P}^n}(\log D)$  entre dans la suite exacte

$$0 \longrightarrow \Omega_{\mathbb{P}^n}^1 \longrightarrow \Omega_{\mathbb{P}^n}(\log D) \longrightarrow \bigoplus_{1 \leq i \leq c} \mathcal{O}_{H_i} \longrightarrow 0$$

qui, en restriction à chaque intersection  $D_I := \bigcap_{i \in I} H_i \cong \mathbb{P}^{n-|I|}$  de dimension non nulle, induit un quotient trivial

$$\Omega_{\mathbb{P}^n}(\log D)|_{D_I} \twoheadrightarrow \mathcal{O}_{D_I}^{\oplus |I|}.$$

Ainsi, on sait déjà que

$$\widetilde{D}_I := \mathbb{P}(\mathcal{O}_{D_I}^{\oplus |I|}) \cong \mathbb{P}^{|I|-1} \times D_I \subset \mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)).$$

On peut se demander si ces sous-variétés de  $\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))$  représentent les seules obstructions à la positivité du fibré cotangent logarithmique. Cela justifie la définition suivante, introduite dans [13].

### Définition 0.3.5

La paire logarithmique  $(\mathbb{P}^n, D)$  a un fibré cotangent logarithmique presque ample si

$$\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)) = \bigcup_{|I| < n} \widetilde{D}_I.$$

On considérera aussi à une propriété légèrement plus faible ([31]).

### Définition 0.3.6

Le fibré cotangent logarithmique est dit ample modulo  $D$  si

$$p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1))) = D.$$

L'intérêt pour ces propriétés provient du fait que de façon plus générale, pour une paire logarithmique lisse  $(X, D)$  quelconque, elles impliquent l'hyperbolicité au sens de Brody du complémentaire  $X \setminus D$ , en vertu du théorème fondamental d'annulation.

Le premier résultat dans cette direction fut obtenu par Noguchi en dimension 2. Bien qu'il n'ait pas utilisé cette terminologie, son résultat peut s'énoncer comme suit.

### Théorème 0.3.7 (Noguchi '86, [86])

Le fibré cotangent logarithmique associé à un arrangement  $D$  de six droites génériques dans  $\mathbb{P}^2$  est ample modulo  $D$ .

Dans [31], ce résultat a été généralisé en dimension supérieure, pour un arrangement  $D$  d'hyperplans dans  $\mathbb{P}^n$ . Les auteurs ont obtenu une borne quadratique en  $n$  sur le nombre d'hyperplans nécessaires pour que le cotangent logarithmique soit ample modulo  $D$ .

**Théorème 0.3.8 (Darondeau-Rousseau '20, [31])**

Le fibré cotangent logarithmique associé à un arrangement de  $c \geq \binom{n+2}{2}$  hyperplans génériques dans  $\mathbb{P}^n$  est ample modulo  $D$ .

Cependant, leur méthode, qui consiste à dégager une certaine propriété de généricité par des calculs d'algèbre linéaire, ne permet pas d'étudier l'optimalité de cette borne.

Le travail que nous présentons dans cette thèse améliore doublement ce résultat : en effet, nous obtenons cette fois une borne linéaire en  $n$  sur le nombre d'hyperplans (liée là aussi à une certaine condition de généricité), et nous montrons qu'elle est optimale, en donnant une description complète du lieu base augmenté.

Fixons un arrangement  $D = H_1 + \dots + H_c$  d'hyperplans en position générale dans  $\mathbb{P}^n$ . On montrera le résultat suivant.

**Théorème 0.3.9**

Le fibré  $\Omega_{\mathbb{P}^n}(\log D)$  est ample modulo  $D$  si, et seulement si, les  $H_i$  ne sont pas tangents à une même surface réglée de degré 2 (une surface quadrique dans un sous-espace linéaire  $\mathbb{P}^3 \subset \mathbb{P}^n$ ).

Par conséquent, le fibré cotangent logarithmique associé à un arrangement de  $c \geq 4n - 2$  hyperplans génériques est ample modulo  $D$ .

Présentons brièvement la stratégie utilisée. Dans un premier temps, on montre que l'image

$$p\left(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1))\right)$$

du lieu base augmenté s'écrit comme une réunion de droites.

En utilisant un résultat de Nakamaye, il suffit alors de caractériser les droites  $l$  en restriction auxquelles le fibré cotangent logarithmique ne sera pas ample. Comme le fibré cotangent logarithmique est globalement engendré, cela revient à l'existence d'un quotient trivial

$$\Omega_{\mathbb{P}^n}(\log D)|_l \twoheadrightarrow \mathcal{O}_l,$$

ou en d'autres termes, d'une section globale

$$\xi \in H^0(l, T_{\mathbb{P}^n}(-\log D)|_l)$$

ne s'annulant nulle part.

On montre alors comment, à partir d'un tel champ de vecteurs, on peut construire une surface quadrique tangente simultanément à  $l$  et à tous les hyperplans  $H_i$ , et réciproquement. Pour ce faire, on forme d'abord l'hypersurface duale dans  $\mathbb{P}^{n*}$  contenant à la fois le  $(n-2)$ -plan  $l^*$  et les points  $H_i^*$ , par une construction dite "de Steiner".

Comme conséquence de ce résultat, on peut décrire exhaustivement le lieu base augmenté, ou du moins sa projection.

**Théorème 0.3.10**

L'image

$$p\left(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1))\right)$$

du lieu base augmenté est la réunion de toutes les surfaces quadriques réglées qui sont

tangentes simultanément à tous les hyperplans  $H_i$ . Il s'ensuit que

$$p\left(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}^n(\log D)}(1))\right) = \mathbb{P}^n$$

dès lors que  $c \leq 3n - 1$ .

Nous terminons cette partie en nous intéressant au fibré cotangent orbifold associé à une paire orbifolde  $(\mathbb{P}^n, \Delta = \sum_{i=1}^c (1 - 1/m_i)H_i)$ , les  $H_i$  étant toujours des hyperplans en position générale, et  $m_i \in \mathbb{N}_{\geq 1} \cup \{+\infty\}$ .

En construisant explicitement des formes symétriques orbifoldes appropriées, on montre que

$$\mathbb{B}_+(\Omega_{(\mathbb{P}^n, \Delta)}) \subset p\left(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}^n(\log D)}(1))\right).$$

De cette inclusion découle la caractérisation suivante.

### **Théorème 0.3.11**

Si les hyperplans vérifient la condition du théorème 0.3.9, et si pour tout  $i$ ,  $m_i \geq 2n$ , alors le fibré cotangent orbifold  $\Omega_{(\mathbb{P}^n, \Delta)}$  est ample modulo  $|\Delta|$ .

Ce résultat permet de redémontrer, par une méthode complètement différente, un cas particulier d'un théorème de Nochka sur l'hyperbolicité de telles paires orbifoldes.

En outre, nous obtenons en corollaire l'hyperbolicité de certaines hypersurfaces de Fermat.



## Notations et conventions

On travaillera toujours sur le corps  $\mathbb{C}$  des nombres complexes.

- $\mathbb{P}^n$  est l'espace projectif complexe de dimension  $n$ .
- La plupart du temps,  $X$  désignera une variété projective lisse connexe de dimension  $n$ .
- Si  $D = \sum_i a_i D_i$  est un  $\mathbb{Q}$ -diviseur dans  $X$ , on pose  $[D] := \sum_i [a_i] D_i$  et  $\lfloor D \rfloor := \sum_i \lfloor a_i \rfloor D_i$ .
- $TX$  et  $\Omega_X = TX^*$  désignent respectivement les fibrés tangent et cotangent holomorphes de  $X$ .
- $K_X = \det(\Omega_X)$  est le fibré canonique de  $X$ .
- En général,  $E$  sera un fibré vectoriel sur  $X$ .
- $\mathbb{P}_{sub}(E)$  est le fibré projectif des **droites** de  $E$ .
- $\mathbb{P}(E) = \mathbb{P}_{sub}(E^\vee) \xrightarrow{\pi_E} X$  est le fibré projectif des **quotients de rang 1** de  $E$ , et  $\mathcal{O}_{\mathbb{P}(E)}(1)$  est le fibré tautologique sur  $\mathbb{P}(E)$ .
- $c_i(E)$  est la  $i$ -ème classe de Chern de  $E$ .
- Si  $L$  est un fibré en droites sur  $X$ , son lieu base

$$\bigcap_{s \in H^0(X, L)} \{s = 0\}$$

est l'ensemble des points de  $X$  annulant toutes les sections de  $E$ .

- On définit un opérateur par  $d^c = \frac{i}{4\pi}(\bar{\partial} - \partial)$ . On a alors  $dd^c = \frac{i}{2\pi}\partial\bar{\partial}$ .
- $\omega_{FS} = \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}$  est la forme de Fubini-Study, c'est-à-dire la forme de Kähler canonique sur  $P^1(\mathbb{C})$ .
- Si  $x \in \mathbb{R}$ , on note  $x^+ = \max\{x, 0\}$ .

Les termes *générique* et *général* sont souvent ambigus en géométrie algébrique. Dans cette thèse, on utilisera les conventions ci-après.

Étant donnée une famille  $\mathcal{X} \rightarrow T$ , nous dirons qu'une propriété  $\mathcal{P}$  est vérifiée par un membre *générique* s'il existe un ouvert de Zariski  $U \subset T$  au-dessus duquel  $\mathcal{P}$  soit vraie. En particulier, une propriété est vérifiée par une hypersurface générale de degré  $d$  si elle l'est sur un ouvert  $U \subset \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ .

De plus, des hypersurfaces  $D_i \subset \mathbb{P}^n$ ,  $i \in I$  seront dites en *position générale* si le diviseur  $D = \sum_{i \in I} D_i$  est à croisements normaux simples.



# Introduction

## 1 Nevanlinna theory

For a rational function  $f$ , the fundamental theorem of algebra gives an estimate of the number of zeroes (resp. poles) of  $f - a$ , for any  $a \in \mathbb{P}^1$ : it is bounded above by the degree of  $f$ . There is plenty of results concerning geometrical properties of these singular points, depending on  $\deg(f)$  and the coefficients of  $f$ . The first results concerning more general meromorphic functions are due to Picard and Borel: namely, Picard's Little Theorem asserts that any non constant holomorphic function  $f : \mathbb{C} \rightarrow \mathbb{CP}^1$  omits at most two values. In the early 20<sup>th</sup> century, following the works of Picard and Borel, R. Nevanlinna evolved after his paper [82] a theory of value distribution for meromorphic functions, which now bears his name. The point of this theory is to compare asymptotically several quantities, which generalise the notions of degree and number of singular points for rational functions. The development of the theory in several complex variables then quickly followed with works of Bloch, Cartan, Weyl-Weyl, and the study of holomorphic curves inside quasi-projective varieties is still a very active research area. Another extension of the theory is to allow complex spaces other than  $\mathbb{C}$  for the source, e.g.  $\mathbb{C}^n$ , parabolic manifolds: this subject has been investigated notably by Ahlfors [1], Stoll [120], Wong, Kobayashi, Carlson-Griffiths, Griffiths-King, Paun-Sibony [93].

Let  $X$  be a smooth projective variety of dimension  $n$ ,  $D \subset X$  a divisor and  $(L, h)$  a Hermitian line bundle on  $X$ . Consider a holomorphic map  $f : \mathbb{C} \rightarrow X$  such that  $f(\mathbb{C}) \not\subset D$ , and write  $f^*D = \sum_{i \in I} n_i(a_i)$ , with  $a_i \in \mathbb{C}$ .

Define the characteristic function of  $L$  by

$$T_{f,L}(r) = \int_1^r \frac{dt}{t} \int_{\Delta_t} f^* c_1(L, h),$$

where  $\Delta_t$  the disk of radius  $t$  centred at 0. This definition is independent of the choice of the metric  $h$ , up to a bounded term (which does not matter for our purpose, since we will search for asymptotic estimates).

The *counting function truncated at order  $k$*

$$N_{f,D}^{[k]}(r) = \int_1^r \frac{dt}{t} \sum_{a_i \in \Delta_t} \max\{n_i, k\}$$

quantifies the average size of  $f(\Delta_t) \cap D$ .

Most of the results in Nevanlinna theory consist in comparing such characteristic functions and obtaining asymptotic estimates to get informations about the entire curves in  $X$ .

Namely, one of the main issues in Nevanlinna theory is to obtain a *Second Main Theorem*-type inequality as follows.

**Conjecture 0.1.1** ([88], **Conj. 4.10.2**) *Let  $X$  be a smooth projective variety,  $D \subset X$  a divisor with simple normal crossings, and  $A$  an ample line bundle on  $X$ . Then there exists  $k_0 \in \mathbb{N}$  such that for any Zariski-dense entire curve  $f : \mathbb{C} \rightarrow X$ , and for arbitrary small  $\varepsilon > 0$  the following inequality holds outside a Borel subset of finite Lebesgue measure:*

$$T_{f, \mathcal{O}_X(D)}(r) + T_{f, K_X}(r) \leq N_{f, D}^{[k_0]}(r) + \varepsilon T_{f, \mathcal{O}_X(D)}(r).$$

An even more optimistic version has been formulated by Griffiths.

**Conjecture 0.1.2 (Griffiths)** *Under the same assumptions as in 0.1.1,*

$$T_{f, \mathcal{O}_X(D)}(r) + T_{f, K_X}(r) \leq N_{f, D}(r) + \varepsilon T_{f, \mathcal{O}_X(D)}(r).$$

## 2 Complex hyperbolicity

The notion of hyperbolicity of complex manifolds goes back to the famous *Little Picard's Theorem* in complex analysis.

### Theorem 0.2.1

*If  $f : \mathbb{C} \rightarrow \mathbb{P}^1$  is a holomorphic map, then  $f$  omits at most two points. Consequently, for any distinct points  $p_1, p_2, p_3$  of  $\mathbb{P}^1$ , there is no non constant holomorphic map  $\mathbb{C} \rightarrow \mathbb{P}^1 \setminus \{p_1, p_2, p_3\}$ .*

In general, one can obtain many information on the geometry of a complex variety  $X$  by understanding the curves inside it. The study of rational curves plays a crucial role in the birational classification of algebraic varieties. When  $X$  is not algebraic, one can look instead at *entire curves*, the non-constant holomorphic maps  $f : \mathbb{C} \rightarrow X$ . One can ask natural questions concerning these entire curves.

1. Are there any entire curves at all in  $X$  ?
2. Is there a proper algebraic subset  $Z \subset X$  containing all entire curves  $f : \mathbb{C} \rightarrow X$  ?
3. Given an entire curve  $f$  in  $X$ , is its image  $f(\mathbb{C})$  Zariski-dense ?

### Definition 0.2.2 ([11])

*A smooth complex variety  $X$  is said hyperbolic in the sense of Brody if it does not contain any entire curve.*

Using this terminology,  $\mathbb{P}^1 \setminus \{p_1, p_2, p_3\}$  is Brody-hyperbolic, while  $\mathbb{P}^1 \setminus \{p_1, p_2\}$  is not (consider the exponential).

If  $X$  satisfies the second point, we shall say that  $X$  is Brody-hyperbolic modulo  $Z$ .

Determining whether a variety is hyperbolic or not is in general a very difficult problem. The main conjecture in hyperbolicity theory was proposed by Green-Griffiths and Lang.

It predicts that a strong hyperbolicity property (point 2 above) holds for a large class of varieties, the so-called *varieties of general type*.

The minimal model program asserts that the geometry of a complex variety  $X$  is mainly described by its canonical bundle  $K_X$ . Recall that a compact variety is of *general type* if its canonical bundle is big, *i.e.* if the dimension of spaces of pluricanonical sections has maximal growth:  $h^0(X, K_X^{\otimes m}) \geq Cm^n$  for some positive constant  $C$ . Little is known concerning the geometry of varieties of general type. According to Green-Griffiths-Lang's conjecture below, one should have a strong control on their entire curves.

**Conjecture 0.2.3 (Green-Griffiths-Lang)** *Let  $X$  be a smooth projective variety of general type. Then there exists a proper algebraic subvariety  $Z \subsetneq X$  containing all entire curves  $f : \mathbb{C} \rightarrow X$ .*

In particular, it would imply that a smooth projective variety  $X$  is hyperbolic if and only if  $X$  and all its subvarieties are of general type.

Remark that this conjecture concerns in particular elliptic and rational curves in  $X$ . For an algebraic surface of general type, the subvariety  $Z$  would have dimension  $\leq 1$ , and consequently there would be only finitely many elliptic or rational curves. Even this small consequence is still conjectural.

The Green-Griffiths-Lang conjecture 0.2.3 would imply a weaker one, stated by Lang.

**Conjecture 0.2.4 ([73])** *Let  $X$  be a smooth projective variety of general type. Then there exists a proper algebraic subset  $Z \subset X$  containing all proper subvarieties (even singular) of  $X$  which are not of general type.*

We will also be interested in the following logarithmic version of this conjecture (for quasi-projective varieties).

**Conjecture 0.2.5** *Let  $X$  be a smooth projective variety and  $D \subset X$  a simple normal crossings divisor. Then the two assertions below are equivalent:*

- *The smooth log-pair  $(X, D)$  is of log-general type, that is,  $K_X(D)$  is a big line bundle.*
- *There exists a proper subvariety  $Z \subset X$  containing all entire curves  $f : \mathbb{C} \rightarrow X$  such that  $f(\mathbb{C}) \not\subset D$ .*

Although some results have been obtained, mostly concerning abelian varieties, certain smooth complete intersections, and hypersurfaces of sufficiently high degree, the two previous conjectures are still widely open.

In [70], Kobayashi used the Poincaré metric on the unit disk to construct a smooth pseudo-metric on any smooth variety  $X$ .

### Definition 0.2.6

*The Kobayashi pseudo-distance  $d_X$  on  $X$  is defined as the largest pseudo-metric such that any holomorphic map  $f : (\Delta, d_P) \rightarrow (X, d_X)$  is distance-decreasing, *i.e.**

$$\forall x, y, d_X(f(x), f(y)) \leq d_P(x, y).$$

**Definition 0.2.7**

A smooth manifold  $X$  is said Kobayashi-hyperbolic if the pseudo-distance  $d_X$  is everywhere non-degenerate.

In Riemannian geometry, the term *hyperbolic* refers to manifolds endowed with a metric of negative curvature. There is actually a reason for this proximity of terminology.

One can show [70] that a complex manifold  $X$  endowed with a smooth hermitian metric with holomorphic sectional curvature bounded above by a constant  $-\gamma < 0$  is Brody-hyperbolic. In particular, in dimension 1, this is an equivalence, and the hyperbolic Riemann surfaces are precisely those uniformised by the unit disk (the negatively curved metric is then the one induced by the Poincaré metric).

One can see easily that the Kobayashi pseudo-distance on  $\mathbb{C}$  vanishes identically. It follows that any Kobayashi-hyperbolic manifold is also hyperbolic in the sense of Brody. The converse does not hold in general, but according to Brody's reparametrisation lemma [11], it is true in the compact case.

### 3 Contents

We now summarise the contents of the thesis and give the main results and ideas. We will always work over the field  $\mathbb{C}$  of complex numbers. Unless otherwise specified,  $X$  denotes a smooth projective complex variety of dimension  $n \geq 1$ .

#### Chapter 1

The first chapter collects various definitions and preliminary results this thesis.

We begin with some properties of hermitian pseudo-metrics defined on complex manifolds. In particular, we give some results on curvature currents and plurisubharmonic functions.

In a second section, we remind the main positivity notions for holomorphic line bundles and the principal characterisations of these properties. We will encounter ample, nef and big line bundles. We recall then how to extend those definitions to vector bundles of higher rank. We define base loci of line bundles and explain how those are related to positivity. After that, we introduce the main definitions concerning complex hyperbolicity.

#### Chapter 2

The second chapter is devoted to *Nevanlinna theoretical* part of this thesis. It opens with a brief account of Nevanlinna's original work for meromorphic functions [82].

#### Value distribution for entire curves

We explain next how the language of Nevanlinna theory has been used to describe the distribution of entire curves in complex varieties.

As before, ‘Second Main Theorem’-type estimations involving the Nevanlinna functions are the key issues of the theory, as such estimations constrain the position of entire curves in the variety.

We mention the principal known examples of Second Main Theorems and describe some applications of this kind of results.

### Value distribution on parabolic manifolds

In the third section, we broaden the questions of Nevanlinna theory to the framework of parabolic complex manifolds. A parabolic manifold is a complex manifold  $M$  endowed with a smooth exhaustion function  $\sigma : M \rightarrow \mathbb{R}_+$  such that  $\log \sigma$  is plurisubharmonic and solution of the homogeneous complex Monge-Ampère equation. The  $(1, 1)$ -form  $dd^c \sigma$  then defines a smooth hermitian pseudo-metric on  $M \setminus \sigma^{-1}(\{0\})$ .

This class includes in particular affine manifolds, finite holomorphic coverings of  $\mathbb{C}^m$ , and non-compact Riemann surfaces having no non-constant bounded above subharmonic function. Following Stoll [121], the concepts of Nevanlinna theory can be well-defined in the parabolic framework.

Most results have been obtained in the particular cases of parabolic Riemann surfaces (Paūn-Sibony ’14 [93], Brotbek-Brunebarbe ’20 [12]), affine manifolds (Gasbarri-Pacienza-Rousseau ’13 [54], Griffiths-King ’73 [63], Carlson-Griffiths ’72 [22], Cornalba-Griffiths ’75 [27], finite coverings of  $\mathbb{C}^m$  (Aihara-Noguchi ’91 [2]) or when the source parabolic manifold has larger dimension than the target (Stoll [121]). In this thesis, we extend some of these results to the general setting. Namely, we obtain a Tautological inequality for holomorphic mappings  $f : M \rightarrow X$  with  $M$  a parabolic manifold of dimension  $m \leq n = \dim X$ . This generalises the Tautological inequalities of [129] and [54].

#### Theorem 0.3.1

Let  $X$  be a smooth projective variety of dimension  $n$ ,  $D \subset X$  a simple normal crossings divisor, and  $f : (M, \sigma) \rightarrow X$  a holomorphic map of maximal rank from an  $m$ -dimensional parabolic manifold  $M$  to  $X$  with  $m \leq n$ . Let

$$f_1 : M \rightarrow X_1 := \mathbb{P} \left( \bigwedge^m \Omega_X(\log D) \right)$$

be the natural meromorphic lifting of  $f$ . One has

$$T_{f_1, \mathcal{O}_{X_1}(1)}(r) \leq_{\text{exc}} N_{f, D}^{[1]}(r) - \text{Ric}_\sigma(r) + O(\log T_{f, \omega}(r)).$$

Furthermore, we prove a Second Main Theorem-type estimation under the assumption that the target variety  $X$  is endowed with a hermitian metric of holomorphic sectional curvature bounded from above by a negative constant.

#### Theorem 0.3.2

Let  $X$  be a smooth projective variety of dimension  $n$  and  $D$  a divisor on  $X$ .

Endow  $X \setminus D$  with a smooth hermitian pseudo-metric  $h$  whose holomorphic sec-

tional curvature is bounded above by a negative constant  $-\gamma$  and holomorphic bisectional curvature is non positive. Assume that the degeneracy set  $\Sigma_h$  of  $h$  is thin.

Let  $(M, \sigma)$  be a parabolic manifold of dimension  $m \leq n$  and  $f : M \rightarrow X$  a non-constant holomorphic map such that  $f(M) \not\subset D$  and  $f(M) \not\subset \Sigma_h$ . Denote by  $\omega$  the  $(1, 1)$ -form associated to the pseudo-metric  $f^*h$  on  $M \setminus f^{-1}(D)_{red}$ . Then  $\omega$  is locally integrable on  $M$  and we have the following second theorem type inequality:

$$\gamma T_{[\omega]}(r) \leq_{exc} 2\pi N_{f,D}^{[1]}(r) - \text{Ric}_\sigma(r) + O(\log T_{[\omega]}(r) + \log r).$$

### Chapter 3

In this chapter, we give the basic material of Campana's theory of geometric orbifolds. We introduce the notion of geometric orbifold pair  $(X, \Delta)$ , which in some sense interpolates between the compact and logarithmic cases. We shall see how these objects can model a ramified covering of a compact variety. We explain afterwards the construction by means of suitable ramified covers of sheaves of orbifold symmetric forms. These sheaves should make sense of formal symmetric forms with fractional poles.

Most features of logarithmic pairs can be extended to the orbifold case. We are naturally interested in orbifold curves, i.e. morphisms  $f : \mathbb{C} \rightarrow X$  ramifying above  $\text{Supp } \Delta$  with the appropriate multiplicity. Therefore, many questions and concepts in hyperbolicity can be transposed to the orbifold setting.

### Chapter 4

In the last chapter, we present some results on complements of hyperplane arrangements  $D = H_1 + \dots + H_c$  in general position in the projective space  $\mathbb{P}^n$ . Recall that this complement is already known to be hyperbolic and hyperbolically embedded in  $\mathbb{P}^n$  when  $c \geq 2n + 1$ . It is known that the distribution of entire curves in  $\mathbb{P}^n$  is controlled by the augmented base locus

$$p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)) \subset \mathbb{P}^n.$$

Though we already know when  $\mathbb{P}^n \setminus D$  is hyperbolic, it is natural to investigate further this augmented base locus.

Note first that the logarithmic cotangent bundle is never ample when  $D \neq \emptyset$ , so that  $\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)) \neq \emptyset$ . More precisely, the bundle  $\Omega_{\mathbb{P}^n}(\log D)$  fits in the exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n}^1 \longrightarrow \Omega_{\mathbb{P}^n}(\log D) \longrightarrow \bigoplus_{1 \leq i \leq c} \mathcal{O}_{H_i} \longrightarrow 0$$

which induces a trivial quotient

$$\Omega_{\mathbb{P}^n}(\log D)|_{D_I} \twoheadrightarrow \mathcal{O}_{D_I}^{\oplus |I|}$$

in restriction to any intersection  $D_I := \bigcap_{i \in I} H_i \cong \mathbb{P}^{n-|I|}$ .



Thus we already know that

$$\widetilde{D}_I := \mathbb{P}(\mathcal{O}_{D_I}^{\oplus |I|}) \cong \mathbb{P}^{|I|-1} \times D_I \subset \mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)).$$

One can ask whether those subvarieties of  $\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))$  are or not the only obstructions to the positivity of the logarithmic cotangent bundle. This motivates the following definition, introduced in [13].

**Definition 0.3.3**

*The log pair  $(\mathbb{P}^n, D)$  has an almost ample logarithmic cotangent bundle if*

$$\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)) = \bigcup_{|I| < n} \widetilde{D}_I.$$

We will also consider a slightly weaker property.

**Definition 0.3.4**

*The logarithmic cotangent bundle is said ample modulo  $D$  if*

$$p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1))) = D.$$

The first result in this direction was obtained by Noguchi in dimension 2. With our terminology, his result can be stated as follows.

**Theorem 0.3.5** ( [86] )

*The logarithmic cotangent bundle associated to a generic arrangement  $D$  of 6 lines in  $\mathbb{P}^2$  is ample modulo  $D$ .*

In the paper [31], this result was generalised to higher dimensions. The authors obtained a quadratic estimation for the minimal number of hyperplanes in order to have amplitude modulo  $D$ .

**Theorem 0.3.6** ( [31] )

*The logarithmic cotangent bundle associated to a generic arrangement of  $c \geq \binom{n+2}{2}$  hyperplanes in  $\mathbb{P}^n$  is ample modulo  $D$ .*

However, they were not able to study the optimality of this bound.

The work we present improves this result on two sides: we obtain a linear bound in the dimension  $n$  for the number of hyperplanes, and show that it is optimal. We also give a complete description of the augmented base locus when it is not minimal.

Set an arrangement  $D = H_1 + \cdots + H_c$  of hyperplanes in general position in  $\mathbb{P}^n$ . We shall prove the following result.

**Theorem 0.3.7**

*The bundle  $\Omega_{\mathbb{P}^n}(\log D)$  is ample modulo  $D$  if, and only if, the hyperplanes  $H_i$  are not tangent to a same ruled quadric surface (equivalently, a quadric surface contained in a linear subspace  $\mathbb{P}^3 \subset \mathbb{P}^n$ ).*

*Consequently, the logarithmic cotangent bundle associated to an arrangement of*

**$c \geq 4n - 2$  generic hyperplanes is ample modulo  $D$ .**

Let us present briefly the strategy of the proof. Firstly, we show that the projection

$$p \left( \mathbb{B}_+ (\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)) \right)$$

is linear (it is a union of lines).

Using a result due to Nakamaye, it is thus enough to characterise the lines  $l$  such that the restriction  $\Omega_{\mathbb{P}^n}(\log D)|_l$  is not ample. As the logarithmic cotangent bundle is globally generated, this amounts to the existence of a trivial quotient

$$\Omega_{\mathbb{P}^n}(\log D)|_l \rightarrow \mathcal{O}_l,$$

or in other words, a nowhere-vanishing global section

$$\xi \in H^0(l, T_{\mathbb{P}^n}(-\log D)|_l).$$

We then show how, starting from such a vector field, we can construct a ruled quadric surface simultaneously tangent to both  $l$  and the hyperplanes  $H_i$  and conversely. Namely, we use ‘Steiner’-type constructions in the dual projective space to get a sufficiently degenerate quadric hypersurface. As a consequence of this result, we can fully describe the projection of the augmented base locus.

**Theorem 0.3.8**

*The projection*

$$p \left( \mathbb{B}_+ (\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)) \right)$$

*is the union of all ruled quadric surfaces which are simultaneously tangent to all hyperplanes  $H_i$ . Hence*

$$p \left( \mathbb{B}_+ (\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)) \right) = \mathbb{P}^n$$

*if  $c \leq 3n - 1$ .*

## Notations and conventions

We will always work over the field  $\mathbb{C}$  of complex numbers.

- $\mathbb{P}^n$  is the  $n$ -th dimensional projective space over  $\mathbb{C}$ .
- In general,  $X$  will denote a smooth connected projective variety of dimension  $n$ , *i.e.* realisable as a closed embedding in some  $\mathbb{P}^N(\mathbb{C})$ .
- If  $D = \sum_i a_i D_i$  is a  $\mathbb{Q}$ -divisor in  $X$ , we define its *round-up*  $\lceil D \rceil := \sum_i \lceil a_i \rceil D_i$  and its *round-down*  $\lfloor D \rfloor := \sum_i \lfloor a_i \rfloor D_i$ .
- $TX$  and  $\Omega_X = TX^*$  denote respectively the holomorphic tangent and cotangent bundles of  $X$ .
- $K_X = \det(\Omega_X)$  is the canonical bundle.
- Usually,  $E$  is a vector bundle over  $X$ .
- $\mathbb{P}_{sub}(E)$  is the projective bundle of **lines** of  $E$ .
- $\mathbb{P}(E) = \mathbb{P}_{sub}(E^\vee)$  denotes the projective bundle of **rank-one quotients** of  $E$ , and  $\mathcal{O}_{\mathbb{P}(E)}(1)$  is the tautological line bundle on  $\mathbb{P}(E)$ .
- $c_i(E)$  is the  $i$ -th Chern class of  $E$ .
- If  $L$  is a line bundle on  $X$ , its base locus is the set

$$\bigcap_{s \in H^0(X, L)} \{s = 0\}$$

of points of  $X$  at which all global sections of  $L$  vanish. The *stable base locus* of  $L$  is the set

$$\mathbb{B}(L) := \bigcap_{m > 0} \text{Bs}(L^{\otimes m}).$$

The *augmented base locus* of  $L$  is the set

$$\mathbb{B}_+(L) := \bigcap_{N \geq 1} \mathbb{B}(L^{\otimes N} \otimes A^{-1})$$

for some ample line bundle  $A$ .

- One defines the operator  $d^c = \frac{i}{4\pi}(\bar{\partial} - \partial)$ . One has then  $dd^c = \frac{i}{2\pi}\partial\bar{\partial}$ .
- $\omega_{FS} = \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}$  is the Fubini-Study form, *i.e.* the canonical Kähler  $(1, 1)$ -form on  $P^1(\mathbb{C})$ .
- For  $x \in \mathbb{R}$ ,  $x^+ = \max\{x, 0\}$ .

Given a family  $\mathcal{X} \rightarrow T$ , we shall say that a property holds for a *general* or *generic* member of this family if there is a Zariski-open subset  $U \subset T$  such that for all  $t \in U$ , the fibre  $X_t$  above  $t$  satisfies this property. In particular, a property holds for a general hypersurface of degree  $d$  in  $\mathbb{P}^n$  if it is satisfied by any element of an open subset  $U \subset |\mathcal{O}_{\mathbb{P}^n}(d)|$ .



# 1

## Preliminaries

In this first chapter, we go over classical concepts and facts that will be used throughout the thesis. The theory of complex hyperbolicity being at the intersection of the algebraic, analytic and differential viewpoints of complex geometry, we will require tools from both of these fields.

### 1.1 Differential geometry of complex manifolds

#### 1.1.1 Metrics and curvature

Let  $X$  be a smooth complex variety of dimension  $n$ . A *smooth hermitian pseudo-metric* on  $X$  is a smooth map  $h : T_X \times_X T_X \rightarrow \mathbb{C}$  such that for any  $x \in X$ , the restricted map

$$h_x : T_{X,x} \times T_{X,x} \rightarrow \mathbb{C}$$

is a non-negative hermitian form. We shall denote by

$$\|\cdot\|_h : T_X \rightarrow \mathbb{R}_+, \xi \mapsto \sqrt{h(\xi, \xi)}$$

the associated pseudo-norm. We define the degeneracy set of  $h$  by

$$\Sigma_h = \{x \in X \mid \exists v \in T_{X,x} \setminus \{0\}, \|v\|_{h,x} = 0\}.$$

Consider holomorphic local coordinates  $(z_1, \dots, z_n)$  defined over an open subset  $U \subset X$ . The pseudo-metric  $h$  has the local expression

$$h(z) = \sum_{i,j} h_{i,j}(z) dz_i d\bar{z}_j,$$

where  $h_{i,j}$  is the smooth function defined by

$$h_{i,j}(z) = h\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right).$$

We associate with  $h$  a global  $(1, 1)$ -form  $\omega_h$ , which can be defined in local coordinates  $(z_1, \dots, z_n)$  by

$$\omega_h = \frac{i}{2} \sum_{i,j} h_{i,j} dz_i \wedge d\bar{z}_j.$$

Let  $\Omega$  be a non-negative smooth  $2n$ -form (pseudo-volume form). On the open set  $U$ , one can define a non-negative smooth real function  $\lambda$  by

$$\Omega = \left(\frac{i}{2}\right)^n \lambda \prod_{1 \leq j \leq n} (dz_j \wedge d\bar{z}_j).$$

The *Ricci curvature* of  $\Omega$  is

$$\text{Ric } \Omega = -2\pi dd^c \log \lambda = -i \sum_{i,j} \frac{\partial^2 \log \lambda}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j.$$

Note that this expression makes sense only on the locus where  $\Omega$  is a positive volume form. Under some conditions, though, it is possible to extend  $\text{Ric } \omega$  as a current on the whole  $X$ .

In particular, if  $\omega$  is the  $(1, 1)$ -form associated to a smooth hermitian pseudo-metric  $h$ , we set  $\text{Ric } \omega := \text{Ric } \omega^n$ .

Let  $D \subset X$  be a divisor and denote by  $\Omega$  a smooth pseudo-volume form defined over the complement  $X \setminus D$ . Let  $U$  be an open subset with holomorphic local coordinates  $z_1, \dots, z_n$  and  $s$  a holomorphic function such that  $D \cap U = (s = 0)$ . As before, let  $\lambda$  be the non-negative smooth function on  $U$  such that  $\Omega = \left(\frac{i}{2}\right)^n \lambda \prod_{1 \leq j \leq n} (dz_j \wedge d\bar{z}_j)$ . We say that  $\lambda$  (or abusively  $\Omega$ ) has *logarithmic singularities* along  $D$  on  $U$  if

$$\lambda(z) = O\left(\frac{1}{|s(z)|^2}\right).$$

If  $\log \lambda$  is locally integrable, it defines a  $(1, 1)$ -current  $[\log \lambda]$ . When the function  $\log \lambda$  is locally integrable in any local chart, one says abusively that  $\log \Omega$  is locally integrable and defines a global  $(1, 1)$ -current on  $X$ , which we call extensively the *Ricci curvature* of  $\Omega$ . Locally, one has

$$\text{Ric}[\Omega] := -2\pi dd^c [\log \lambda].$$

Here the differential is to be understood in the sense of distributions, that is, for any smooth  $(n-1, n-1)$ -form  $\phi$  with compact support,

$$\langle \text{Ric}[\Omega], \phi \rangle = \int (-2\pi \log \lambda) dd^c \phi.$$

### 1.1.2 Plurisubharmonic functions

Plurisubharmonic functions were introduced by Lelong and Oka in order to study holomorphically convex sets. We refer to [76] for more details. They are the higher-dimension generalisation of subharmonic functions over  $\mathbb{C}$ .

Recall that a function  $u : U \rightarrow [-\infty, +\infty[$  defined on an open subset  $U \subset \mathbb{C}$  is *subharmonic* if it satisfies the *mean value inequality*

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta$$

for any  $z \in U$  and  $r > 0$  such that  $r < d(z, \partial U)$ .

**Definition 1.1.1**

A function  $u : \Omega \rightarrow [-\infty, +\infty[$  defined over an open subset  $\Omega \subset \mathbb{C}^n$  is said to be *plurisubharmonic* if:

1.  $u$  is upper-semi-continuous;
2. for every complex line  $\ell \subset \mathbb{C}^m$ , the restriction  $u|_{\Omega \cap \ell}$  is subharmonic. This means that for all  $a \in \Omega$  and  $\xi \in \mathbb{C}^n$  such that  $|\xi| < d(a, \partial\Omega)$ ,

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + e^{i\theta}\xi) d\theta.$$

The space of plurisubharmonic (psh) functions over  $\Omega$  is denoted by  $\text{Psh}(\Omega)$ .

Recall that a function  $v : U \subset \mathbb{C} \rightarrow [-\infty, +\infty[$  is subharmonic if it satisfies the sub-mean value property at any point  $z \in U$ :

$$v(z) \leq \frac{1}{2\pi} \int_0^{2\pi} v(z + re^{i\theta}) d\theta \text{ whenever } D(z, r) \subset U.$$

It follows that property 2 above is equivalent to the fact that  $u$  satisfies the inequality

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + e^{i\theta}\xi) d\theta$$

for all  $z \in \Omega$  and  $\xi \in \mathbb{C}^m$  such that  $|\xi| < d(z, \Omega^c)$ .

Plurisubharmonic functions have several properties.

1. Assume that  $u \not\equiv -\infty$  on every connected component of  $\Omega$ . Then  $u$  is locally integrable on  $\Omega$ .
2. For any decreasing sequence of functions  $u_k \in \text{Psh}(\Omega)$ , the limit  $u = \lim_{k \rightarrow \infty} u_k$  is psh.
3. Let  $u \in \text{Psh}(\Omega)$  such that  $u \not\equiv -\infty$  on any connected component of  $\Omega$ . Assume that  $(\rho_\varepsilon)$  is a family of smoothing kernels. Then the  $u \star \rho_\varepsilon$  are smooth and psh on  $\Omega_\varepsilon = \Omega \cap B(0, \varepsilon)$ , the family  $(u \star \rho_\varepsilon)$  is non-decreasing in  $\varepsilon$  and converges to  $u$  as  $\varepsilon \rightarrow 0$ .
4. Let  $u_1, \dots, u_p \in \text{Psh}(\Omega)$  and  $\chi : \mathbb{R}^p \rightarrow \mathbb{R}$  be a convex function which is non-decreasing in each variable. Then  $\chi(u_1, \dots, u_p)$  is psh. In particular,  $u_1 + \dots + u_p$ ,  $\max(u_1, \dots, u_p)$ ,  $\log(e^{u_1}, \dots, e^{u_p})$  are psh.

If  $u$  is an element of  $\mathcal{C}^2(\Omega, \mathbb{R})$ , then  $u$  is plurisubharmonic if and only if its complex Hessian

$$\left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} (a) \right)_{1 \leq j, k \leq n}$$

is a semi-positive matrix at every point  $a \in \Omega$ . Passing to the limit (with point  $c$ ) above), this is still true for general psh functions.

**Theorem 1.1.2**

Let  $u \in \text{Psh}(\Omega)$  such that  $u \not\equiv -\infty$  on any connected component of  $\Omega$ . For all  $\xi \in \mathbb{C}^n$ , the distribution

$$Hu(\xi) := \sum_{1 \leq j, k \leq n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \xi_j \bar{\xi}_k$$

is a positive measure. Conversely, for any  $v \in \mathcal{D}'(\Omega)$  such that  $Hv(\xi)$  is a positive measure for any  $\xi \in \mathbb{C}^n$ , there exists a unique psh function  $u$  on  $\Omega$  such that  $v$  is the distribution associated to  $u$ .

This notion can naturally be generalised to complex manifolds. Let  $X, Y$  be two complex manifolds. Consider a holomorphic map  $\varphi : X \rightarrow Y$  and a holomorphic function  $u \in \mathcal{C}^2(Y, \mathbb{R})$ . Since  $\partial\bar{\partial}(u \circ \varphi) = \varphi^* \partial\bar{\partial}u$ , we have

$$H(u \circ \varphi)(z, \xi) = Hu(\varphi(z), d\varphi(z) \cdot \xi),$$

so that the Hermitian form  $Hu$  on  $T_X$  is independent of the choice of local coordinates. This allows us to define plurisubharmonic functions on complex manifolds.

A map  $u : M \rightarrow \mathbb{R} \cup \{-\infty\}$  is said *plurisubharmonic* if for any holomorphic chart  $\phi : V \subset \mathbb{C}^m \xrightarrow{\sim} U \subset M$ , the function  $u \circ \phi : V \rightarrow \mathbb{R} \cup \{-\infty\}$  is plurisubharmonic.

We now give two basic results relating plurisubharmonic functions and Ricci currents which will be used further in chapter 2.

**Lemma 1.1.3** *Let  $\psi$  be a plurisubharmonic function on a complex manifold  $X$  which is of class  $\mathcal{C}^\infty$  over  $\{\psi \neq -\infty\}$ . Assume that  $\psi$  is bounded from above. Then  $\psi$  extends as a plurisubharmonic function on  $X$ . Moreover, the  $(1, 1)$ -form  $dd^c\psi$  is locally integrable on  $\Delta$  and one has*

$$[dd^c\psi] \leq dd^c[\psi]$$

in the sense of currents.

**Proof:** The first point is a classical result in the theory of plurisubharmonic functions.

It follows from Theorem 5.24, p. 45 of [36].

Let  $\eta$  be a smooth  $(n-1, n-1)$ -form with compact support in  $X$  and  $(\eta_n)$  a sequence of smooth  $(n-1, n-1)$ -form with compact support in  $X \setminus \{\psi = -\infty\}$  converging to  $\eta$  such that  $0 \leq \eta_n \leq \eta$ . Then  $[dd^c\psi](\eta_n)$  converges to  $[dd^c\psi](\eta)$  by monotone convergence theorem.

Moreover,  $\eta - \eta_n \geq 0$  and  $\psi$  is plurisubharmonic, so that

$$dd^c[\psi](\eta - \eta_n) \geq 0.$$



We obtain

$$[dd^c\psi](\eta_n) = dd^c[\psi](\eta_n) \leq dd^c[\psi](\eta)$$

and conclude by passing to the limit. ■

**Lemma 1.1.4** *Let  $X$  be an  $n$ -dimensional complex manifold with a reduced divisor  $D$ . Let  $h$  be a smooth pseudo-metric defined on  $X \setminus D$  with associated  $(1, 1)$ -form  $\omega$ . Assume that  $\omega^n$  has at most logarithmic singularities along  $D$  and that  $\log \omega^n$  is locally integrable over  $X$ . If there exists a smooth  $d$ -closed  $(1, 1)$ -form  $\tilde{\omega}$  defined on  $X$  such that*

$$\tilde{\omega} - \text{Ric} \omega \geq 0$$

on  $X \setminus (D \cup \Sigma_h)$ , then:

1. both currents  $[\text{Ric} \omega]$  and  $\text{Ric}[\omega]$  are well-defined over  $X$ ;
2. one has

$$[-\text{Ric} \omega] \leq 2\pi[D] + (-\text{Ric}[\omega]).$$

**Proof:** The statement is local, hence we can restrict to an open subset  $U$ . Fix holomorphic local coordinates  $z_1, \dots, z_n$  on  $U$  and a holomorphic function  $s$  such that  $D \cap U$  is defined by  $(s = 0)$ . Let  $\lambda$  be the non-positive smooth function on  $U$  such that  $\omega^n = \left(\frac{i}{2}\right)^n \lambda \prod_{1 \leq j \leq n} (dz_j \wedge d\bar{z}_j)$ .

Define a function by  $\phi(z) = \log(|s(z)|^2 \lambda(z))$ . By assumption,  $\phi$  is bounded from above on  $U$ . Moreover,  $\phi$  is smooth outside  $E = D \cup \Sigma_h$ . Applying the  $dd^c$ -lemma to  $\tilde{\omega}$ , up to taking an open subset of  $U$ , there exists a smooth function  $\alpha$  such that  $\tilde{\omega} = dd^c \alpha$ .

By assumption, the function  $\psi = \phi + \alpha$  is plurisubharmonic and bounded from above. Hence  $dd^c \psi$  is locally integrable by the previous lemma, and so is  $dd^c \phi$  outside  $E$ . This shows that  $[\text{Ric} \omega]$  is well-defined. The lemma also gives

$$[dd^c \alpha] + [dd^c \phi] \leq dd^c[\alpha] + dd^c[\phi].$$

Since  $\alpha$  is  $\mathbb{C}^\infty$ ,  $[dd^c \alpha] = dd^c[\alpha]$ , and Lelong-Poincaré's lemma applied to  $D$  gives

$$[-\text{Ric} \omega] = [dd^c \phi] \leq dd^c[\phi] = dd^c[\log |s|^2] + dd^c[\log \lambda] = [D] - \text{Ric}[\omega].$$
■

## 1.2 Positivity of line bundles

We review the classical positivity notions for line bundles. For further details, we refer to the two reference books from Lazarsfeld [74], [75], and Griffiths-Harris [62].

In the next two sections, unless otherwise specified,  $X$  denotes a smooth projective variety of dimension  $n \geq 1$ .

### 1.2.1 Definitions

Consider a holomorphic line bundle  $L$  over  $X$ . Assume that  $H^0(X, L) \neq 0$ . The evaluation map  $\text{eval} : H^0(X, L) \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow L$  gives rise to a rational map

$$\Phi_L : X \dashrightarrow \mathbb{P}(H^0(X, L))$$

from  $X$  to the projective space of one-dimensional quotients of  $H^0(X, L)$ . Concretely,  $\Phi_L(x)$  is the hyperplane in  $H^0(X, L)$  of sections vanishing at  $x$ .

Choosing a basis  $s_0, \dots, s_N$  of  $H^0(X, L)$ , we identify  $\mathbb{P}(H^0(X, L))$  with the projective space  $\mathbb{P}^N$ . Though abusive, since then  $s_i$  take values in a line bundle, this identification is legitimate, since an element  $[s_0(x) : \dots : s_N(x)]$  is independent of the local trivialisation. Then our map has the local expression

$$\begin{aligned} \Phi_L : X &\dashrightarrow \mathbb{P}^N \\ x &\mapsto [s_0(x) : \dots : s_N(x)] \end{aligned}$$

The element  $[s_0(x) : \dots : s_N(x)]$  of  $\mathbb{P}^N$  is independent of the trivialisation.

#### Definition 1.2.1

*The line bundle is said:*

- very ample if  $\Phi_L$  is an embedding;
- ample if some power  $L^{\otimes m}$  is ample;
- big if  $\Phi_{L^{\otimes m}}$  is birational onto its image, for some  $m \geq 1$ .

A Cartier divisor  $D$  is termed ample/big if the associated line bundle  $\mathcal{O}_X(D)$  is so. More generally, a  $\mathbb{Q}$ - or  $\mathbb{R}$ - Cartier divisor  $D$  is ample/big if it can be written as a linear combination of ample integral divisors with positive coefficients.

Recall that the Néron-Severi group of  $X$  is defined to be the group  $N^1(X) = \text{Div}_{\mathbb{R}}(X) / \equiv_{\text{num}}$  of numerical equivalence classes. It is well-known that  $N^1(X)$  is a finite dimensional  $\mathbb{R}$ -vector space. We then can consider the open cones  $\text{Amp}(X)$  and  $\text{Big}(X)$  in  $N^1(X)$  generated respectively by ample and big divisors. We introduce two other positivity notions corresponding to the closures of these cones.

#### Definition 1.2.2

*A line bundle  $L$  is said*

- nef (for numerically eventually free) if its class  $[L]$  belongs to  $\text{Nef}(X) := \overline{\text{Amp}(X)}$ ,
- pseudoeffective if  $[L] \in \text{Pseff}(X) := \overline{\text{Big}(X)}$

### 1.2.2 Properties

We now gather the main properties of these positivity notions for line bundles. We refer to [74] for details and proofs. A remarkable fact concerning the notion of ampleness is that it has equivalent geometric, numerical, cohomological and metric definitions.

**Theorem 1.2.3 (Cartan-Serre-Grothendieck)**

The following properties are equivalent:

1.  $L$  is ample.
2. For any coherent sheaf  $\mathcal{F}$  on  $X$ , there exists  $m_1 \in \mathbb{N}$  such that for all  $i > 0$ , for all  $k \geq m_1$ ,
 
$$H^i(X, \mathcal{F} \otimes L^k) = 0.$$
3. For any coherent sheaf  $\mathcal{F}$  on  $X$ , there exists  $m_2 \in \mathbb{N}$  such that  $\mathcal{F} \otimes L^k$  is globally generated for  $k \geq m_2$ .
4. There exists  $m_3 \in \mathbb{N}$  such that  $L^{\otimes k}$  is very ample for all  $k \geq m_3$ .

**Theorem 1.2.4 (Nakai-Moishezon-Kleiman criterion)**

A line bundle  $L$  (resp. the divisor  $D$ ) is ample if and only if

$$\int_V c_1(L)^{\dim V} > 0 \quad (\text{resp. } (D^{\dim V} \cdot V) > 0)$$

for any positive-dimensional irreducible subvariety  $V \subseteq X$ .

In fact, this characterisation still holds for  $\mathbb{R}$ -divisors, see [74], Theorem 2.3.18. Interpreting nef line bundles as limits of ample bundles, we can hope for a similar numerical characterisation. In fact, it is even sufficient to check nefness on curves.

**Theorem 1.2.5 (Kleiman's theorem)**

Let  $L$  be a holomorphic line bundle on  $X$ . If  $L$  is nef, then for any irreducible subvariety  $V \subseteq X$ ,

$$\int_V c_1(L)^{\dim V} \geq 0.$$

Similarly, if an  $\mathbb{R}$ -divisor  $D$  is nef, then for any  $V$ ,  $(D^k \cdot V) \geq 0$ .

This result makes nefness a central notion for line bundles, as it is far easier to handle than ampleness.

Ample and nef line bundles behave well with respect to finite maps.

**Proposition 1.2.6**

Let  $f : X \rightarrow Y$  be a finite map of smooth projective varieties and  $L$  a line bundle on  $X$

1. If  $L$  is ample, then  $f^*L$  is ample. In particular, the restriction of  $L$  to any subvariety of  $X$  is ample.
2. If  $f$  is surjective and  $f^*L$  is ample, then  $L$  is ample.

**Proposition 1.2.7**

Let  $L$  be a line bundle on  $X$ ,  $Y$  a smooth projective variety and  $f : Y \rightarrow X$  a proper map.

1. If  $L$  is nef, then  $f^*L$  is a nef line bundle on  $Y$ .
2. If  $f$  is surjective and  $f^*L$  is nef, then  $L$  is nef.

As a consequence, any globally generated line bundle  $L$  is nef.

We have two useful alternative descriptions of big line bundles.

**Proposition 1.2.8**

Denote  $n = \dim X$ . The following are equivalent.

1.  $L$  is big.
2. For some/any ample line bundle  $A$  in  $X$ , there exists a positive integer  $m \geq 1$  and an effective line bundle  $E$  such that  $L^m \simeq A \otimes E$ .
3.  $L$  has maximal Itaka dimension  $\kappa(X, L)$ , i.e. there exists a constant  $C > 0$  such that

$$h^0(X, L^m) \geq Cm^n$$

for any integer  $m$  large and divisible enough.

Let us now describe how ampleness can be interpreted in terms of curvature when one endows  $X$  with a Hermitian metric. This is essential, since it builds a bridge between the algebraic theory of line bundles and geometric properties of complex manifolds.

**Theorem 1.2.9 (Kodaira embedding theorem)**

Let  $X$  be a compact Kähler manifold, and  $L$  a holomorphic line bundle on  $X$ . The following are equivalent.

1.  $L$  is positive, i.e. there exists a smooth Hermitian metric  $h$  on  $L$  with positive curvature form  $\frac{i}{2\pi}\Theta(L, h)$ ;
2. there exists a holomorphic embedding

$$\iota : X \rightarrow \mathbb{P}^N$$

such that  $\iota^*\mathcal{O}_{\mathbb{P}^N}(1) = L^{\otimes m}$  for some  $m > 0$ .

We also have analogous characterisations for nefness and bigness.

Recall that a *singular metric* on a line bundle  $L$  is simply a hermitian metric  $h$  which is given in any trivialisation by a weight of the form  $e^{-\phi}$ , where  $\phi$  is a locally integrable function. We have a well-defined *curvature current*

$$\frac{i}{2\pi}\Theta(L, h) = \frac{i}{2\pi}\partial\bar{\partial}\phi.$$

The case when  $\Theta(L, h)$  is a positive/non-negative current is especially interesting, since it can be related to the positivity of  $L$ .

**Proposition 1.2.10 (Metric characterisations of nef and big line bundles)**

- $L$  is nef if and only if for any  $\varepsilon > 0$ , there exists a smooth Hermitian metric  $h_\varepsilon$  on  $L$  such that

$$\frac{i}{2\pi}\Theta(L, h_\varepsilon) > -\varepsilon\omega$$

where  $\omega$  is some fixed Kähler form on  $X$ .

- $L$  is big if and only if there exists a singular Hermitian metric  $h$  on  $L$  such that

$$\frac{i}{2\pi}\Theta(L, h)$$

is a positive  $(1, 1)$ -current on  $X$ .

For any hermitian line bundle  $(L, h)$  on a smooth (not necessarily projective) complex manifold, Lelong-Poincaré's formula relates a holomorphic section of  $L$ , the integration current along its zero divisor and the curvature form of  $(L, h)$ .

**Theorem 1.2.11 (Lelong-Poincaré's equation)**

Let  $X$  be a smooth complex variety endowed with a hermitian line bundle  $(L, h)$ . Let  $\sigma \in H^0(X, L)$  be a holomorphic global section of  $L$  with zero divisor  $Z$ . The function  $\log \|\sigma\|_h^2$  is locally integrable on  $X \setminus Z$  and one has the following equality of currents:

$$-dd^c[\log \|\sigma\|_h^2] = c_1(L, h) - [Z].$$

In fact, this equation still holds for singular metrics.

### 1.2.3 Base loci

Let  $L$  be a holomorphic line bundle on  $X$ . We denote

$$\text{Bs}(L) = \{x \in X; s(x) = 0 \quad \forall s \in H^0(X, L)\}.$$

**Definition 1.2.12**

The stable base locus of  $L$  is the algebraic set  $\mathbb{B}(L) = \bigcap_{m \geq 1} \text{Bs}(L^{\otimes m})$ . There exists an integer  $m_0$  such that  $\mathbb{B}(L) = \text{Bs}(L^{m_0 k})$  for all  $k \gg 0$ .

**Definition 1.2.13**

The augmented base locus of  $L$  is the algebraic set

$$\mathbb{B}_+(L) = \bigcap_{N \geq 0} \mathbb{B}(L^N \otimes A^{-1}) = \bigcap_{q \geq 0, N \geq 0} \mathbb{B}(L^{Nq} \otimes A^{-q}),$$

for some ample line bundle  $A$  (this definition is independent of the choice of the ample line bundle).

Clearly,  $\mathbb{B}_+(L)$  is empty if and only if  $L$  is ample, and it is a proper subset of  $X$  if and only if  $L$  is big.

The augmented base locus  $\mathbb{B}_+(L)$  can somehow be understood as the locus where  $L$  is not ample. The following results give sense to this connection.

**Theorem 1.2.14 (Boucksom-Cacciola-Lopez '14, [10])**

Let  $L$  be a big line bundle on  $X$ . Then the complement  $X \setminus \mathbb{B}_+(L)$  is the largest Zariski open subset  $U \subseteq X \setminus \mathbb{B}(L)$  such that, for all large and divisible  $m$ , the restriction of the morphism

$$\Phi_m : X \setminus \mathbb{B}(L) \rightarrow \mathbb{P}(H^0(X, L^m))$$

to  $U$  is an isomorphism onto its image.

**Corollary 1.2.15** *Let  $L$  be a globally generated line bundle and  $V \subset X$  a closed subset. If  $\Phi_L$  defines an immersive map on  $X \setminus V$ , then  $\mathbb{B}_+(L) \subseteq V$ .*

**Proof:** Since  $L$  is globally generated, it is *a fortiori* nef. According to [10], Thm. A, the augmented base locus  $\mathbb{B}_+(L)$  is the smallest closed subset  $V$  of  $X$  such that  $\Phi_{L^q}$  is an isomorphism of  $X \setminus V$  onto its image, for  $q$  large enough. By [74], Lemma 2.1.28, for  $q$  large enough, there exists a finite morphism  $\nu_q$  such that we have the Stein factorisation of  $\Phi_L$

$$X \xrightarrow{\Phi_{L^q}} \Phi_{L^q}(X) \xrightarrow{\nu_q} \Phi_L(X).$$

By assumption, the fibres of  $\Phi_L|_{X \setminus V}$  are discrete. It follows that on  $X \setminus V$ ,  $\Phi_{L^q}$  has discrete and connected fibres as well. In other words, for  $q$  large enough, the linear system  $|L^q|$  defines an isomorphism of  $X \setminus V$  onto its image. ■

**Theorem 1.2.16 (Nakamaye)**

*Let  $L$  be a big and nef line bundle on a smooth projective variety  $X$ . Then*

$$\mathbb{B}_+(L) = \bigcup_{\substack{Z \subset X \\ Z \cdot L^{\dim Z} = 0}} Z.$$

## 1.3 Vector bundles

Ampleness is defined for vector bundles so as to extend the notion for line bundles while preserving the cohomological properties.

In what follows,  $X$  is a smooth projective complex variety and  $E$  is a holomorphic vector bundle of rank  $r$  on  $X$ . We consider the projectivised bundle  $\mathbb{P}(E) \xrightarrow{\pi_E} X$ . We denote by  $\mathcal{O}_{\mathbb{P}(E)}(1)$  the tautological line bundle on the projective variety  $\mathbb{P}(E)$ .

**Definition 1.3.1**

*The vector bundle  $E$  is said ample/nef/big if the tautological line bundle  $\mathcal{O}_{\mathbb{P}(E)}(1)$  is so on  $\mathbb{P}(E)$ .*

Since  $(\pi_E)_* \mathcal{O}_{\mathbb{P}(E)}(m) = S^m E$ , many properties of ample line bundles are preserved by replacing the tensor powers with the symmetric powers of  $E$ . In particular, ampleness can still be detected cohomologically.

**Theorem 1.3.2**

*The following properties are equivalent:*

1.  $E$  is ample.
2. For any coherent sheaf  $\mathcal{F}$  on  $X$ , there exists  $m_1 \in \mathbb{N}$  such that for all  $i > 0$ , for all  $k \geq m_1$ ,

$$H^i(X, \mathcal{F} \otimes S^k E) = 0.$$

3. For any coherent sheaf  $\mathcal{F}$  on  $X$ , there exists  $m_2 \in \mathbb{N}$  such that  $\mathcal{F} \otimes S^k E$  is globally generated for  $k \geq m_2$ .
4. Given any ample line bundle  $A$  on  $X$ , there exists  $m_3 \in \mathbb{N}$  such that for all  $k \geq m_3$ ,  $S^k E$  is the quotient of some  $A^{\oplus N}$ .

Ample vector bundles satisfy some important properties which follow directly from the corresponding results on line bundles.

**Proposition 1.3.3**

1. If  $E$  is ample/nef, then so is any quotient bundle of  $E$ .
2. Let  $f : Y \rightarrow X$  be a finite map. If  $E$  is ample/nef, then so is the pullback  $f^*E$  on  $Y$ .
3. Assume moreover that  $f$  is surjective. If  $f^*E$  is ample/nef, then so is  $E$ .

Let  $E$  be a globally generated line bundle on a smooth projective variety  $X$ . There exists a quotient

$$H^0(X, E) \otimes \mathcal{O}_X \rightarrow E \rightarrow 0$$

inducing a morphism

$$\phi : \mathbb{P}(E) \rightarrow \mathbb{P}^{N-1}$$

with  $N = h^0(X, E)$ . By construction,  $\mathcal{O}_{\mathbb{P}(E)}(1) = \phi^* \mathcal{O}_{\mathbb{P}^{N-1}}(1)$ .

Then  $E$  is ample if and only if  $\phi$  is finite.

If the morphism  $\phi$  is not finite, it contracts some curve  $C \in \mathbb{P}(E)$  to a point. This can be reformulated as follows.

**Proposition 1.3.4 (Gieseker's lemma)**

*Suppose that  $E$  is globally generated. Then  $E$  fails to be ample if and only if there exists a curve  $C \subset X$  such that the restriction  $E|_C$  has a trivial quotient.*

## 1.4 Complex hyperbolicity

This section is dedicated to several notions of hyperbolicity for complex varieties. We refer to [70] and [73] for more details.

### 1.4.1 Motivation in dimension one

The theory of complex hyperbolicity trace back to the work of Picard.

**Theorem 1.4.1 (Little Picard theorem)**

*Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be an entire map. If  $f(\mathbb{C})$  omits two distinct points, then  $f$  is constant.*

This property of the space  $\mathbb{C} \setminus \{0, 1\} = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$  is of particular interest, as we will see in this section. This motivates the following definition.

**Definition 1.4.2**

Let  $X$  be a complex space. We say that  $X$  is Brody-hyperbolic if every holomorphic map  $f : \mathbb{C} \rightarrow X$  is constant.

Another fundamental example is the unit disc  $\Delta$ . It even satisfies a stronger property, called *Kobayashi-hyperbolicity*: there exists a smooth hermitian metric, namely the Poincaré metric  $ds_P^2$ , which has negative Chern curvature form and such that any holomorphic map  $h : \Delta \rightarrow \Delta$  is distance decreasing with respect to  $h_P$ . This particular geometry of  $\Delta$  is the starting point in order to extend the notion of Kobayashi-hyperbolicity.

**1.4.2 The Kobayashi pseudo-metric**

Let  $X$  be a complex space of dimension  $n$ .

**Definition 1.4.3**

The Kobayashi-Royden infinitesimal pseudo-metric is the function  $k_X : T_X \rightarrow \mathbb{R}_+$  defined by

$$\begin{aligned} k_X(\xi) &= \inf \{ \lambda \in \mathbb{R}_+; \exists f : \Delta \rightarrow X \text{ holomorphic such that } \lambda f'(0) = \xi \} \\ &= \inf \{ \|u\|_{h_P}; f : \Delta \rightarrow X \text{ holomorphic, } u \in T_\Delta, df(u) = \xi \}. \end{aligned}$$

It is easy to see from the definition that any holomorphic map is decreasing with respect to  $k_X$ . Namely, for any morphism  $f : X \rightarrow Y$  between two complex manifolds, and for any  $\xi \in T_X$ , one has

$$k_X(\xi) \leq k_Y(f_*\xi).$$

**Definition 1.4.4**

The Kobayashi pseudo-metric on  $X$  is the geodesic pseudo-instance obtained by integrating  $k_X$ : for any  $x, y \in X$ , one defines

$$d_X(x, y) = \inf_{\gamma} \int_0^1 k_X(\gamma'(t)) dt$$

where the infimum runs over all piecewise smooth curves  $\gamma : [0, 1] \rightarrow X$  joining  $x$  to  $y$ .

**Proposition 1.4.5**

The Kobayashi pseudo-metric satisfies the distance-decreasing property. It is in fact the largest pseudo-distance on  $X$  with this property, in the following sense: let  $\delta : X \times X \rightarrow \mathbb{R}_+$  be a pseudo-distance such that for any holomorphic map  $f : \Delta \rightarrow X$ , and for any  $w, z \in \Delta$ , one has

$$\delta(f(w), f(z)) \leq d_P(w, z).$$

Then for all  $x, y \in X$ ,

$$\delta(x, y) \leq d_X(x, y).$$

We can now introduce the notion of hyperbolicity.

**Definition 1.4.6**

The complex manifold  $X$  is said Kobayashi-hyperbolic if the Kobayashi pseudo-distance on  $X$  is actually a distance, i.e. if it is non-degenerate.



By definition, the unit disc  $\Delta$  is hyperbolic.

We easily show that Kobayashi- implies Brody-hyperbolicity (it follows from the fact that the Kobayashi pseudo-metric on  $\mathbb{C}$  vanishes identically). When the manifold  $X$  is compact, the two notions are equivalent. This is a consequence of Brody's reparametrisation lemma.

**Lemma 1.4.7 (Brody's reparametrisation lemma)** *Let  $X$  be a complex manifold endowed with a hermitian metric  $h$ , and let  $f : \Delta \rightarrow X$  be a holomorphic map. For any  $0 < r < 1$ , there exists  $R \geq \|f'(0)\|_h$  and a biholomorphism  $\phi : \Delta_R \rightarrow \Delta_r$  satisfying:*

1.  $\|(f \circ \phi)'(0)\|_h = 1$ ;

2. for all  $t \in \Delta_R$ ,

$$\|(f \circ \phi)'(z)_h \leq \frac{R^2}{R^2 - |z|^2}.$$

### 1.4.3 Manifolds with negative holomorphic sectional curvature

The Poincaré-Koebe uniformisation theorem implies that every Riemann surface can be uniformised either by  $\mathbb{C}$ ,  $\mathbb{P}^1(\mathbb{C})$  or  $\Delta$ . It follows that a Riemann surface is hyperbolic if and only if it is uniformised by  $\Delta$ .

#### Proposition 1.4.8

*Let  $X$  be Riemann surface. If  $X$  admits a metric  $h$  with curvature bounded by a negative constant, then it is hyperbolic. In particular, a compact Riemann surface of genus  $\geq 2$  is hyperbolic.*

We can generalise these considerations to higher dimensional manifolds.

#### Definition 1.4.9

*Let  $X$  be a complex manifold endowed with a hermitian metric  $h$ . The holomorphic sectional curvature of  $h$  is the function  $\text{HSC}_h : \mathbb{P}_{\text{sub}}(T_X) \rightarrow \mathbb{R}$  defined by*

$$\text{HSC}_h(x, [\xi]) = \sup_{f: \Delta \rightarrow X} K_{f^*h}(0),$$

*where the supremum is taken over all holomorphic maps  $f : \Delta \rightarrow X$  such that  $f(0) = x$  and such that  $f'(0) \in \mathbb{C} \cdot \xi$ .*

#### Proposition 1.4.10

*Let  $X$  be a complex manifold endowed with a hermitian metric  $h$  whose sectional curvature is bounded from above by a negative constant  $-\gamma < 0$ . Then  $X$  is hyperbolic in the sense of Kobayashi.*

### 1.4.4 Hyperbolic embeddings

#### Definition 1.4.11

Let  $X$  be a complex manifold, and  $U \subset X$  a dense open subset. We say that  $U$  is hyperbolically embedded in  $X$  if for any  $x \in X$  and any open neighbourhood  $V_1$  of  $x$ , there exists another open neighbourhood  $V_2$  of  $x$  such that  $\overline{V_2} \subset V_1$  and

$$d_U(U \cap V_1, U \cap V_2) > 0.$$

#### Proposition 1.4.12

Let  $X$  be a compact complex manifold and  $U$  a dense open subset. The following are equivalent:

1.  $U$  is hyperbolically embedded in  $X$ .
2. For any hermitian metric  $h$  on  $X$ , there exists  $\varepsilon > 0$  such that

$$\varepsilon \|\cdot\|_h|_U \leq F_U.$$

### 1.4.5 Criteria for hyperbolicity

The theorem that makes the link between these bundles and hyperbolicity is the following.

#### Theorem 1.4.13 (Fundamental vanishing theorem)

Let  $A$  be an ample line bundle on a smooth projective variety  $X$ . Then for any entire curve  $f : \mathbb{C} \rightarrow X$  and all global symmetric differential form

$$\omega \in H^0(X, S^m \Omega_X \otimes A^{-1})$$

one has  $f^* \omega \equiv 0$ .

This has a logarithmic version.

#### Theorem 1.4.14 (Fundamental vanishing theorem, logarithmic case)

Let  $(X, D)$  be a smooth logarithmic pair and  $A$  an ample line bundle on  $X$ . Then for any entire curve  $f : \mathbb{C} \rightarrow X \setminus D$  and all global symmetric logarithmic form

$$\omega \in H^0(X, S^m \Omega_X(\log D) \otimes A^{-1})$$

one has  $f^* \omega \equiv 0$ .

These results give some information on the geometry of the Green-Griffiths locus, i.e. the closure of the union of images of all entire curves  $f : \mathbb{C} \rightarrow X$  (resp.  $X \setminus D$ ). Namely, it implies that this closed subset of  $X$  is entirely contained in the image inside  $X$  of the augmented base locus

$$\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_X)}(1)) \text{ (resp. } \mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_X(\log D))}(1)) \text{)}.$$

Many authors contributed to prove this theorem and its generalisation to jet differentials: Noguchi [84], Green-Griffiths [57], Demailly [33], Siu-Yeung [117].

This theorem is essential, as it provided one of the main techniques in complex hyperbolicity: to control the location of entire curves, one has to construct enough explicit global symmetric forms (or jet differentials) vanishing on an ample divisor and determine their common vanishing locus.

Few general criteria for hyperbolicity are known up to now. In particular, we have the following.

**Theorem 1.4.15 (Kobayashi, [71])**

*Let  $X$  be a smooth projective variety. If the cotangent bundle  $\Omega_X$  is ample, then  $X$  is Brody-hyperbolic (hence also Kobayashi-hyperbolic).*

Indeed, assume that  $\Omega_X$  is ample, and take some ample line bundle  $A$ . Then there is some  $m \geq 1$  such that the bundle  $S^m \Omega_X \otimes A^{-1}$  is globally generated. One concludes with the vanishing theorem applied to sections  $P \in H^0(X, S^m \Omega_X \otimes A^{-1})$ . This generalises to the logarithmic case as follows.

**Theorem 1.4.16 ([70])**

*Let  $X$  be a smooth projective variety and  $D \subset X$  a simple normal crossings divisor. Assume that the logarithmic cotangent bundle  $\Omega_X(\log D)$  is ample modulo  $D$ . Then the complement  $X \setminus D$  is Brody-hyperbolic.*

If we ask for finer properties of the log-pair  $(X, D)$ , we get that the complement is even hyperbolically embedded, by a result of Green.

**Theorem 1.4.17 ([70])**

*Let  $X$  be a projective variety and  $D = \sum_{1 \leq i \leq r} D_i$  a divisor in  $X$ . Assume that both  $D$  and  $X \setminus D$  are Brody-hyperbolic. Then  $X \setminus D$  is Kobayashi-hyperbolic.*

*If in addition, for each  $\emptyset \neq I \subset \{1, \dots, r\}$ , the stratum  $\bigcap_{i \in I} D_i \setminus \left( \bigcup_{j \notin I} D_j \right)$  is Brody-hyperbolic, then  $X \setminus D$  is hyperbolically embedded in  $X$ .*

Zaidenberg derived from this the following result.

**Theorem 1.4.18 ([70])**

*Let  $X$  be a smooth projective variety and  $D = \sum_{i=1}^c D_i$  a divisor with simple normal crossings. Let  $D^{(k)}$ ,  $1 \leq k \leq n$  denote the stratum of  $D$  consisting of the points of  $D$  of multiplicity  $k$ :*

$$D^{(k)} = D^k \setminus D^{k+1}, \text{ where } D^k = \bigcup_{1 \leq i_1 < \dots < i_k \leq c} D_{i_1} \cap \dots \cap D_{i_k}.$$

*Then the complement  $U = X \setminus D$  is hyperbolically embedded in  $X$  if and only if both  $U$  and all the strata  $D^{(k)}$  are Brody-hyperbolic.*

**Theorem 1.4.19 (Bloch-Ochiai)**

*Let  $X$  be a smooth projective algebraic variety. If the irregularity  $h^1(X, \Omega_X)$  is larger than  $\dim X$ , then every entire curve  $f : \mathbb{C} \rightarrow X$  is algebraically degenerate.*

**Corollary 1.4.20** *A non-singular algebraic surface with irregularity  $> 2$  is Brody-hyperbolic if, and only if, it does not contain curves of genus 0 or 1.*



## 2

# Nevanlinna theory

For a rational function  $f$ , the fundamental theorem of algebra gives an estimate of the number of zeroes (resp. poles) of  $f - a$ , for any  $a \in \mathbb{P}^1$ : it is bounded above by the degree of  $f$ . There are lots of other results concerning geometrical properties of these singular points, depending on  $\deg(f)$  and the coefficients of  $f$ . The first results concerning more general meromorphic functions are due to Picard and Borel: namely, Picard's Little Theorem asserts that any non constant holomorphic function  $f : \mathbb{C} \rightarrow \mathbb{CP}^1$  omits at most two values. In the early 20<sup>th</sup> century, following the works of Picard and Borel, R. Nevanlinna evolved after his paper [82] a theory of value distribution for meromorphic functions, which now bears his name. The point of this theory is to compare asymptotically several quantities, which generalise the notions of degree and number of singular points for rational functions. The development of the theory in several complex variables then quickly followed with works by Bloch, Cartan, Weyl-Weyl, and the study of holomorphic curves inside quasi-projective varieties is still a very active research area. Another extension of the theory is to allow complex spaces other than  $\mathbb{C}$  for the source, e.g.  $\mathbb{C}^n$ , parabolic manifolds: this subject has been investigated notably by Ahlfors [1], Stoll [120], Wong [130], Kobayashi, Carlson-Griffiths [22, 23], Cornalba-Griffiths [27], Griffiths-King [63], Paun-Sibony [93] et al.

The core of Nevanlinna theory consists of three properties, called the three Main Theorems of Nevanlinna (the Third Main Theorem can also be called the Defect Relation). Consider a holomorphic map  $f : M \rightarrow X$  from a (non-compact) complex manifold  $M$  to a smooth projective variety  $X$ . The First and Second Main Theorems provide bounds on the growth of the mapping  $f$  in terms of the distance of the image of  $f$  to a given divisor  $D$ . More precisely, the First Main Theorem essentially says that the size of the preimage set of any point  $a \in X$  is asymptotically independent of  $a$ .

We will first give an overview of Nevanlinna's original work on value distribution of meromorphic functions. In a second section, we will extend the theory to entire curves  $f : \mathbb{C} \rightarrow X$  inside a complex projective variety  $X$ .

## 2.1 In one variable

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a meromorphic function, that is, a holomorphic map from  $\mathbb{C}$  to  $\mathbb{P}^1(\mathbb{C})$ .

For a divisor  $E = \sum_i m_i(a_i)$  on  $\mathbb{C}$ , define the *truncated counting functions to level  $k$*  by

$$n_k(t, E) = \sum_{|a_i| < t} \min\{k, m_i\}, t > 0$$

$$N_k(r, E) = \int_1^r n_k(t, E) \frac{dt}{t}, r > 1.$$

In particular,  $n = n_\infty$  and  $N = N_\infty$  are simply called *counting functions*.

For any point  $a \in \mathbb{P}^1(\mathbb{C})$ , one wants to estimate the asymptotic number  $n(r, (f - a)_0)$  of solutions (counted with multiplicity) of the equation

$$f(z) = a$$

inside the open disk  $\Delta_r = \Delta(0, r)$ , as  $r \rightarrow \infty$ . Here  $(f - a)_0$  is the zero divisor of  $f - a$ . The *counting function*

$$N(r, (f - a)) = \int_1^r n(f, a, t) \frac{dt}{t}$$

computes the average number of preimages of  $a$  inside the disks centred at 0. One can interpret it as the growth of the size of the preimage set of  $a$ .

The *proximity function*

$$m(r, f, a) = \int_{|z|=r} \log^+ \left( \frac{1}{|f(z) - a|} \right) dz$$

, describes how close is  $f$  to  $a$  over the circle  $\partial\Delta_r$ . One indeed sees that  $m(f, a, r)$  is big when  $\|f - a\|_{L^1(\Delta_r)}$  is small.

We begin with the fundamental Jensen's formula. Let  $\phi : \mathbb{C} \rightarrow [-\infty, +\infty]$  be a function such that the singular set  $\{\phi = \pm\infty\}$  is a discrete set denoted by  $S = \{a_i, i \geq 1\}$ . Assume that  $\phi$  is of class  $\mathcal{C}^2$  on  $\mathbb{C} \setminus S$  and has logarithmic singularities at the points of  $S$ : this means that

$$\phi(z) = \lambda_i \log |z - a_i| + g_i(z)$$

in a neighbourhood of each  $a_i$ , for some  $\lambda_i \in \mathbb{R}$  and  $\psi_i$  of class  $\mathcal{C}^2$ .

**Lemma 2.1.1 (Jensen's formula)** *Let  $\phi$  be as above, and assume that  $\phi(0) \neq \pm\infty$ . Then for every  $0 < s < r$ ,*

$$\frac{1}{2\pi} \int_0^{2\pi} \phi(re^{i\theta}) d\theta - \frac{1}{2\pi} \int_0^{2\pi} \phi(se^{i\theta}) d\theta = 2 \int_s^r \frac{dt}{t} \int_{\Delta_t} \frac{i}{2\pi} \partial\bar{\partial}\phi + \int_s^r \frac{dt}{t} \sum_{|a_i| < t} \lambda_i.$$

The *First Main Theorem of Nevanlinna* states that as  $r$  goes to infinity, the sum  $N(f, a, r) + m(f, a, r)$  is essentially independent of the point  $a$  (up to a bounded term, which may depend on  $a$ ), and is related in some way to the *growth* of  $f$ . More precisely, define the *characteristic function* by

$$T(r, f) = N(r, (f)_\infty) + m(r, f)$$

where we denote by  $(f)_\infty$  the divisor of poles of  $f$ . Then we have

**Theorem 2.1.2 (First Main Theorem)**

For any point  $a \in \mathbb{P}^1(\mathbb{C})$ , one has

$$T\left(r, \frac{1}{f-a}\right) = T(r, f) + O(1),$$

where  $O(1)$  denotes a bounded term (possibly depending on  $a$  and  $f$ ).

**Proof:** We apply Jensen's formula with  $\phi(z) = \log |f(z) - a|$ . Outside  $\text{Supp}(\text{div}(f - a))$ , one has  $dd^c\phi(z) \cong 0$ , so that

$$N(r, \text{div}(f - a)) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta - \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta$$

Since  $(f - a)_\infty = (f)_\infty$ , we get

$$N(r, (f-a)_0) + m\left(r, \frac{1}{f-a}\right) = N(r, (f)_\infty) + \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(e^{i\theta})| d\theta - \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta.$$

The formula result follows then from the inequality  $|\log^+ |f(z) - a| - \log^+ |f(z)|| \leq \log^+ |a| + \log 2$ . ■

As a consequence, we derive a bound on the order function.

**Proposition 2.1.3 (Nevanlinna's inequality)**

There exists a constant  $C$  such that for each  $a \in \mathbb{P}^1$ ,

$$N(r, (f - a)_0) \leq T(r, f) + C.$$

This characteristic function has a geometric interpretation in terms of the growth of the function  $f$ . Denote by  $\omega_{FS} = dd^c \log(1 + |w|^2)$  the Fubini-Study form on  $\mathbb{P}^1(\mathbb{C})$ . We introduce *Ahlfors-Shimizu's order function*

$$T_f(r, \omega_{FS}) = \int_1^r \frac{dt}{t} \int_{\Delta_r} f^* \omega_{FS} = \int_1^r \frac{dt}{t} \int_{\Delta_r} dd^c \log(1 + |f(z)|^2).$$

This function quantifies the average size of the image  $f(\Delta_t)$  of the disk by  $f$ . In fact, we have

$$T_f(r, \omega_{FS}) = T(r, f) + O(1).$$

We also have *Crofton's formula*

$$T_f(r, \omega_{FS}) = \int_{w \in \mathbb{P}^1(\mathbb{C})} N(r, (f - w)_0) \omega_{FS}.$$

The function  $T(r, f)$  can be seen as a generalisation of the degree in the rational case. Indeed, one computes for example

- when  $f = P/Q$  is a rational function with  $P, Q$  coprime,

$$T(r, f) = \max\{\deg P, \deg Q\} \log r + O(1);$$

- for any  $p \in \mathbb{N}$ ,  $T(r, e^{z^p}) = \frac{r^p}{\pi}$ .

The characteristic function gives a criterion for rationality.

**Proposition 2.1.4**

A meromorphic function  $f$  is rational iff  $T(r, f) = \mathcal{O}(\log r)$ .

### 2.1.1 The Second Main Theorem

The Second Main Theorem of Nevanlinna is a result opposite to the previous ones, since it gives an estimate from above of the characteristic function using counting functions. We introduce a specific notation. For two non-negative functions  $\phi, \psi$  defined on an interval  $[r_0, +\infty[$ ,  $r_0 \geq 0$ , we write

$$\phi(r) \leq_{\text{exc}} \psi(r)$$

if the inequality holds outside a Borel subset of finite Lebesgue measure. We begin by a basic calculus lemma.

**Lemma 2.1.5 (Borel)** *Let  $\phi(r)$ ,  $r \geq r_0 \geq 0$  be a non-negative, monotone increasing function. Then for every  $\delta > 0$ ,*

$$\frac{d}{dr} \phi(r) \leq_{\text{exc}} \phi(r)^{1+\delta}.$$

Here the Borel subset outside which the inequality does not hold depends on  $\delta$ .

**Lemma 2.1.6 (Lemma of logarithmic derivative)** *Let  $f$  be a meromorphic function. Then for any  $\delta > 0$ ,*

$$m(r, f'/f) \leq_{\text{exc}} \left(1 + \frac{(1+\delta)^2}{2}\right) \log^+ T(r, f) + S(r, f)$$

where  $S(r, f) = O(\log T(r, f) + \log r)$  is a small term.

#### Theorem 2.1.7

*Let  $f$  be a meromorphic function, and let  $a_1, \dots, a_q$  be distinct points of  $\mathbb{P}^1$ . Then*

$$(q-2)T(r, f) \leq_{\text{exc}} \sum_{i=1}^q N^{[1]}(r, (f - a_i)_0) + S(r, f).$$

## 2.2 Nevanlinna theory for holomorphic curves inside projective varieties

This section is devoted to the extension of Nevanlinna theory to holomorphic curves inside a projective variety. We will explain for instance how to obtain hyperbolicity results via Second main theorem type inequalities. We fix a smooth projective variety  $X$  of dimension  $n$  and a holomorphic map  $f : \mathbb{C} \rightarrow X$ .

Similarly to the one-dimensional case, we shall define several functions to describe value distribution of  $f$ . Let  $D \subset X$  be an effective divisor such that  $f(\mathbb{C}) \not\subset \text{Supp } D$ . Write  $f^*D = \sum_{i \in I} m_i(a_i)$  for some points  $a_i \in \mathbb{C}$  and integers  $m_i \geq 1$ . Consider a smooth Hermitian metric  $h_D$  on the line bundle  $\mathcal{O}(D)$  and a section  $s_D$  of  $\mathcal{O}(D)$  such that  $D = (s_D = 0)$ .

The (*integrated*) *counting function* is defined by

$$N_{f,D}(r) = \int_1^r \frac{dt}{t} \left( \sum_{i \in I; |a_i| < t} m_i \right).$$



The *truncated counting function* at order  $k \geq 1$  is

$$N_{f,D}^{[k]}(r) = \int_1^r \frac{dt}{t} \left( \sum_{i \in I; |a_i| < t} \min(k, m_i) \right).$$

The *proximity function*  $m_{f,D}$  is given by

$$m_{f,D}(r) = \int_{\partial\Delta_r} -\log \|s \circ f\|_{h_D} dw$$

Let  $(L, h)$  a Hermitian holomorphic line bundle on  $X$ . The *characteristic function* of  $L$  is the function

$$T_{f,L}(r) = \int_1^r \frac{dt}{t} \int_{\Delta_t} f^* c_1(L, h).$$

Up to a bounded term (which happens to be irrelevant for our considerations), this expression is indeed independent of the metric  $h$ .

### 2.2.1 The First Main Theorem

As before, we infer from Jensen's formula that the sum  $N_{f,D}(r) + m_{f,D}(r)$  essentially depends only on the linear system  $|\mathcal{O}_X(D)|$ . This is the First Main Theorem:

#### Theorem 2.2.1

*With the previous notation, one has*

$$N_{f,D}(r) + m_{f,D}(r) = T_{f,\mathcal{O}_X(D)}(r) + O(1).$$

As a consequence, one obtains *Nevanlinna's inequality*:

$$N_{f,D}(r) \leq T_{f,\mathcal{O}_X(D)}(r) + O(1).$$

### 2.2.2 Second Main Theorem

Let  $L$  be a line bundle on  $X$ . In the opposite direction, one would like to bound from above the characteristic function  $T_{f,L}$  by the counting function  $N_{f,D}$  of an effective divisor  $D$ . Namely, one wants to obtain a *Second Main Theorem (SMT)* for a pair  $(X, D)$  and a line bundle  $L$ , that is, an inequality of the form

$$T_{f,L}(r) \leq_{\text{exc}} \alpha N_{f,D}^{[k]} + S_f(r),$$

where  $S_f(r)$  is an error term of the form  $O(\log r + \log T_{f,A}(r))$  for some ample line bundle  $A$ .

A result of this type is known in only very few cases, mainly when  $X = \mathbb{P}^n$  or a semi-abelian variety. We refer to [88] for a large panorama of the developments regarding the Second Main Theorem.

**Theorem 2.2.2 (Cartan-Nochka's SMT)**

Let  $f : \mathbb{C} \rightarrow \mathbb{P}^n(\mathbb{C})$  be an entire curve and let  $m$  be the dimension of the smallest linear subspace  $V$  containing  $f(\mathbb{C})$ . Let  $H_i \not\supset V$ ,  $1 \leq i \leq q$  be hyperplanes in general position. Then

$$(q - 2n + m - 1)T_{f, \mathcal{O}_{\mathbb{P}^n}(1)}(r) \leq_{exc} \sum_{i=1}^q N_{f, H_i}^{[m]}(r) + S_f(r).$$

**Corollary 2.2.3** Under the same condition, the following hold.

1. Defect relation:  $\sum_{j=1}^q \delta_n(f, H_j) \leq 2n - m + 1$ .
2. Ramification theorem: For each  $j = 1, \dots, q$ , assume that  $f$  ramifies at order  $\mu_j$  over  $H_j$ . Then

$$\sum_{j=1}^q \left(1 - \frac{n}{\mu_j}\right) \leq 2n - m + 1.$$

**Remark 2.2.4** With the orbifold terminology, we consider the orbifold pair  $(\mathbb{P}^n, \Delta = \sum_{j=1}^q (1 - 1/\mu_j)H_j)$ . The Ramification theorem amounts to saying that if  $\mu_j \geq n$  for each  $j$ , then for any orbifold curve  $f$  whose image is included in an  $m$ -dimensional linear subspace,

$$\deg(\mathbb{K}_{\mathbb{P}^n} + \Delta^{(n)}) \leq n - m.$$

## 2.2.3 Some open problems and applications

### Diophantine approximation and Nevanlinna theory

Since the 80's, with works of Vojta, it is known that some number-theoretic questions (known as Diophantine approximation theory) could be related to Nevanlinna theory, the most famous being Masser-Oesterlé 'abc-conjecture'.

**Conjecture 2.2.5 (Masser-Oesterlé)** Let  $\varepsilon > 0$ . Then there exists a constant  $C_\varepsilon > 0$  such that for all integers  $a, b, c \in \mathbb{Z}$  with  $a + b + c = 0$  and  $\gcd(a, b, c) = 1$ , one has

$$\max\{|a|, |b|, |c|\} \leq C \prod_{p \text{ prime}; p|abc} p^{1+\varepsilon}.$$

It is shown by Vojta [129] that 2.2.5 would be a consequence of an analogue of the Second Main Theorem for projective varieties defined over number fields.

### Relation to hyperbolicity

A crucial reason to seek for 'Second main theorems' is that inequalities of this form can give a lot of information on the degeneracy of holomorphic curves. In particular, if a strong enough Second main theorem holds for a pair  $(X, D)$  of log-general type, then  $(X, D)$  also satisfies Green-Griffiths' conjecture. Namely, one hopes that the following conjecture of Griffiths is true.

**Conjecture 2.2.6** (see [88], Conj. 4.10.5) *Let  $X$  be a smooth projective variety,  $D \subset X$  a simple normal crossings divisor and  $A$  an ample line bundle on  $X$ . Then for any Zariski-dense entire curve  $f : \mathbb{C} \rightarrow X$  and for all  $\varepsilon > 0$  the following Second main theorem holds.*

$$T_{f, K_X(D)} \leq_{exc} N_{f,D}(r) + \varepsilon T_{f,A}(r).$$

This conjecture would imply a weak version of the logarithmic Green-Griffiths conjecture: if  $(X, D)$  is of log-general type, then some tensor power of  $K_X(D)$  decomposes as  $K_X^{\otimes m} = A \otimes \mathcal{O}(E)$ , for some ample line bundle  $A$  and effective divisor  $E$ . Any non-degenerate entire curve  $f : \mathbb{C} \rightarrow X \setminus D$  would satisfy

$$T_{f, \mathcal{O}(E)}(r) < 0$$

outside a Borel subset of finite Lebesgue measure, which is impossible.

In the opposite direction, it is natural to ask whether these two conjectures are equivalent, and try to establish a Second Main Theorem when one already knows that a pair  $(X, D)$  satisfies Green-Griffiths-Lang conjecture.

## 2.3 Value distribution on parabolic spaces

Consider the function  $\sigma : z \in \mathbb{C} \mapsto |z| \in \mathbb{R}_+$ . This is an *exhaustion* of  $\mathbb{C}$ :  $\sigma$  is unbounded and  $\sigma^{-1}([0, r]) = \overline{\Delta}_r$  is compact for any  $r \geq 0$ . Moreover, the function  $\log \sigma$  is harmonic. The pair  $(\mathbb{C}, \sigma)$  is what we call a parabolic complex manifold. In this section, we generalise this concept and give properties and examples of parabolic spaces. For more details and proofs, we refer to the standard book on this subject written by Wilhelm Stoll, *Value distribution on Parabolic Spaces* [121].

### 2.3.1 Definitions

Let  $M$  be a complex manifold of dimension  $m > 0$ . An exhaustion of  $M$  is a proper smooth map  $\sigma : M \rightarrow \mathbb{R}_+$ . We define

- the respectively closed and open pseudoballs of radius  $r$

$$M[r] = \{z \in M | \sigma(z) \leq r\} = \sigma^{-1}([0, r]) \text{ and } M\langle r \rangle = \sigma^{-1}([0, r));$$

- the pseudo-sphere  $M\langle r \rangle = M[r] \setminus M(r)$ ;
- $M[s, r] = M[r] \setminus M(s)$  if  $0 \leq s \leq r$
- $r_0 = \min\{r \in \mathbb{R}_+ | M[r] \neq \emptyset\}$ .

Given  $A \subset M$ , we denote  $A[r] = A \cap M[r]$  and do similarly for the other sets.

The exhaustion  $\sigma$  is

- *central* if  $M[0]$  has measure zero;
- *strong* if  $(dd^c \sigma)^m \not\equiv 0$  on  $M(r)$  for any  $r > r_0$ ;
- (*strictly*) *pseudo-convex* if  $dd^c \sigma \geq 0 (> 0)$
- *logarithmic pseudo-convex* if  $dd^c \log \sigma \geq 0$  on  $M \setminus M[0]$ .

A central and logarithmic pseudo-convex exhaustion is *parabolic* if  $(dd^c \sigma)^m \not\equiv 0$  and  $(dd^c \log \sigma)^m \equiv 0$  on  $M^+ = M \setminus M[0]$ . It is said *strict* if  $dd^c \sigma > 0$  on  $M$ .

Such a pair  $(M, \sigma)$  with  $\sigma$  parabolic is called a parabolic manifold.

In order to simplify the equations, we define:

$$\alpha = dd^c \sigma \qquad \beta = dd^c \log \sigma \qquad (2.1)$$

$$\eta = d^c \log \sigma \wedge (dd^c \log \sigma)^{m-1} \qquad \nu = d^c \sigma \wedge (dd^c \sigma)^{m-1} \qquad (2.2)$$

$$\eta_r = \eta|_{M\langle r \rangle} \qquad \nu_r = \nu|_{M\langle r \rangle} \qquad (2.3)$$

**Lemma 2.3.1** *The following relations are satisfied on  $M^+$ :*

$$d\nu = \alpha^m \qquad d\eta = \beta^m = 0 \qquad \sigma^m \eta = \nu \qquad (2.4)$$

$$\sigma \alpha^m = m d\sigma \wedge \nu = m \sigma^m d\sigma \wedge \eta \qquad (2.5)$$

$$\zeta = \int_{M\langle r \rangle} \eta_r \text{ is constant} \qquad (2.6)$$

$$\int_{M\langle r \rangle} \alpha^m = \int_{M\langle r \rangle} \nu_r = r^{2m} \zeta \qquad (2.7)$$

**Example 2.3.2**

Explicit examples of parabolic spaces are difficult to construct but we shall give some basic ones.

1. The most standard example is  $(\mathbb{C}^m, \sigma)$  where  $\sigma(z) = |z|^2$  for all  $z$ . One has  $dd^c \sigma = \frac{i}{2\pi} \sum_j dz_j \wedge d\bar{z}_j > 0$  and directly  $dd^c \log \sigma = 0$  on  $\mathbb{C}^m \setminus \{0\}$ . In this case, the pseudo-balls are simply the Euclidean balls.
2. Further, let now  $\beta : M \rightarrow \mathbb{C}^m$  be a holomorphic covering of  $\mathbb{C}^m$  by a complex manifold  $M$  (that is, a proper surjective holomorphic map of strict rank  $m$ ). Then  $\beta^* \sigma$  is a parabolic exhaustion on  $M$ , where  $\sigma(z) = |z|^2$ . Moreover,  $\zeta$  equals the sheet number of the covering.
3. In particular, any affine variety is parabolic.
4. A non-compact Riemann surface is parabolic, if and only if every bounded above subharmonic function is constant.
5. An non-affine example of a higher-dimensional parabolic manifold is given by

$$(\mathbb{C} \setminus \mathbb{Z}) \times \mathbb{C}^m$$

(cf Stoll [121], p. 85).

We can also find in Stoll's book the following interesting example.

**Proposition 2.3.3** (see Prop. 10.15 in [121])

Let  $X$  be a compact complex manifold of dimension  $m - 1$  and  $L \rightarrow X$  a non-positive holomorphic line bundle over  $X$  endowed with a smooth hermitian metric  $\kappa$  such that  $\omega = c_1(L, \kappa) \leq 0$ . Assume that there exists a non-empty open subset  $U \subset X$  on which  $c_1(L, \kappa) < 0$ . Then the function defined on the total space  $M$  of  $L$  by  $\sigma(z) = |z|_\kappa^2$  is a parabolic exhaustion with

$$\int_X \omega^{m-1} > 0.$$

**Proof:** It is straightforward to see that  $\sigma$  is a central exhaustion with  $dd^c \log \sigma = \pi^* \omega \geq 0$ , hence logarithmic pseudo-convex. Moreover, since  $\dim X = m - 1$ , one has  $\omega^m = 0$ , so that  $(dd^c \log \sigma)^m = 0$ . Define then  $\phi = \pi|_{B(1)}$  and  $j : \Delta(0, 1) \subset \mathbb{C} \rightarrow \phi^{-1}(\phi(x_0))$  by  $j(z) = zx_0$ , for some fixed  $x_0 \in S(1)$ . Clearly  $j$  is biholomorphic. ■

**2.3.2 Jensen's formula**

Consider a parabolic manifold  $(M, \sigma)$  of dimension  $m$ .

We have the following generalisation of Jensen's formula.

**Proposition 2.3.4** (Thm. 11.2, [121])

Let  $\phi : M \rightarrow [-\infty, +\infty[$  be a function which can locally on  $M$  be written as a difference of two plurisubharmonic functions. Then we have for  $r > r_0$ ,

$$\int_{r_0}^r \frac{dt}{t} \int_{M(t)} dd^c \log \phi \wedge \beta^{m-1} = \int_{M\langle r \rangle} \phi \eta_r - \int_{M\langle r_0 \rangle} \phi \eta_{r_0} = \int_{M\langle r \rangle} \phi \eta_r + O(1).$$

**Proof:** To start with, remark that we can assume that  $\phi$  is smooth (otherwise we can use a partition of unity). For each regular value  $r$  of  $\sigma$ , the function  $\phi$  is integrable with respect to the measure  $\eta_r$  over the pseudo-sphere  $M\langle r \rangle$ . We have

$$\begin{aligned} \int_{r_0}^r \frac{dt}{t} \int_{M(t)} dd^c \phi \wedge \beta^{m-1} &= \int_{r_0}^r \frac{dt}{t} \int_{M(r)} \mathbf{1}_{M(t)} dd^c \phi \wedge \beta^{m-1} \\ &= \int_{M(r)} \int_{\sigma(x)}^r \frac{dt}{t} dd^c \phi \wedge \beta^{m-1} \\ &= \int_{M(r)} \log \frac{r}{\sigma} dd^c \phi \wedge \beta^{m-1} \\ &= - \int_{M(r)} \phi dd^c \log \sigma \wedge \beta^{m-1} + \int_{M(r)} (\phi d^c \log \sigma - \log(r/\sigma) d^c \phi) \wedge \beta^{m-1} \blacksquare \end{aligned}$$

## 2.4 Holomorphic maps from a parabolic manifold

We fix once and for all an  $m$ -dimensional parabolic manifold  $(M, \sigma)$ .

Throughout the rest of this chapter, we shall present several results on value distribution theory of holomorphic mappings  $f : M \rightarrow X$  into a smooth projective variety  $X$ . Each time, we will mention what had already been obtained either in the case of parabolic Riemann surfaces or  $\mathbb{C}^n$ .

### 2.4.1 Definitions

In this section, we provide a summary of the main objects and basic results of value distribution theory for parabolic manifolds. For a more detailed presentation, we mainly refer to [121], [122], [123], [108], [88], [2], [93] and [12].

If  $\beta$  is a current of type  $(1, 1)$  and order 0 on  $M$ , we define the *characteristic function* of  $\beta$  for  $r > r_0$  by

$$T_\beta(r) = \int_{r_0}^r \frac{dt}{t^{2m-1}} \int_{M(t)} \beta \wedge \alpha^{m-1}.$$

If  $\Sigma$  is an analytic  $(n-1, n-1)$ -cycle in  $M$ , one has a well-defined integration current  $[\Sigma]$  and we define the *counting function with respect to  $\Sigma$*  by  $N_\Sigma = T_{[\Sigma]}$ .

Let  $f : M \rightarrow X$  be a holomorphic map from  $M$  to some projective variety  $X$ . Consider a holomorphic line bundle  $L$  on  $X$  endowed with a Hermitian metric  $h$ , and a divisor  $D$  on  $X$  such that  $f(M) \not\subset \text{Supp } D$ , and write  $(f^*D)_{red} = \sum_i a_i D_i$ . Choose a section  $s \in H^0(X, \mathcal{O}_X(D))$  such that  $D = \text{div}(s)$  and a Hermitian metric  $h_D$  on  $\mathcal{O}_X(D)$ .

We let for  $q \geq 1$  and  $r > r_0$  :

$$\begin{aligned} T_{f,L}(r) &= T_{f^*c_1(L,h)}(r) \\ N_{f,D}^{[q]}(r) &= T_{D^{[q]}}(r), \text{ where } D^{[q]} = \sum_i \min(a_i, q) D_i \\ m_{f,D}(r) &= - \int_{M\langle r \rangle} \log(|s \circ f|_{h_D}) \eta_r \geq 0 \end{aligned}$$

It is not difficult to see that the definitions of these functions are independent of the choice of metrics, up to a bounded term.

The function  $m_{f,D}$  is the *proximity function*: it evaluates precisely how the image of  $f$  approaches the divisor  $D$  on the pseudo-sphere of radius  $r$ ;  $m_{f,D}$  goes to infinity as  $f(M)$  gets closer to  $D$  on  $M\langle r \rangle$ .

As in the case  $M = \mathbb{C}$ , these functions satisfy the First main theorem.

**Theorem 2.4.1** ( [121], Theorem 12.3)

*In the previous context, let  $L$  be a holomorphic line bundle on  $X$  and  $D$  a divisor in the linear system  $|L|$ . Then for all  $r > r_0$ ,*

$$T_{f,L}(r) = N_{f,D}(r) + m_{f,D}(r) + O(1).$$

### 2.4.2 Lemma of logarithmic derivative

Let us recall a standard notation from Nevanlinna theory. Given two functions  $g, h : [r_0, +\infty) \rightarrow \mathbb{R}$  we write

$$g(r) \leq_{\text{exc}} h(r)$$

if there exists a Borel subset  $E \subset [r_0, +\infty)$  of finite Lebesgue measure such that  $g(r) \leq h(r)$  for all  $r \in [r_0, +\infty) \setminus E$ . Moreover, given  $s : [r_0, +\infty) \rightarrow \mathbb{R}$ , we shall write  $g(t) \leq_{\text{exc}} h(t) + O(s(t))$  if there exists  $C \in \mathbb{R}_+$  such that  $g(t) \leq_{\text{exc}} h(t) + Cs(t)$ . We will use several times the following well-known calculus result:

**Proposition 2.4.2 (Borel's lemma)**

*Let  $\phi : [r_0, +\infty[ \rightarrow \mathbb{R}$  be a monotone increasing function.*

*Then for all  $\epsilon > 0$ , one has*

$$\phi'(r) \leq_{\text{exc}} \phi(r)^{1+\epsilon}.$$

**Proof:** Since  $\phi$  is monotone, we know that  $\phi$  is differentiable almost everywhere.

Assume  $\phi \not\equiv 0$  and take  $r_1 \geq r_0$  such that  $\phi(r_1) > 0$ . Set

$$E(\epsilon) = \{\phi'(r) > \phi(r)^{1+\epsilon}\}.$$

Then the Lebesgue measure of  $E(\epsilon)$  equals

$$\begin{aligned} \int_{E(\epsilon)} 1 dr &\leq \int_{E(\epsilon)} \frac{\phi'(r)}{\phi(r)^{1+\epsilon}} \\ &\leq \int_{r_1}^{+\infty} \frac{\phi'(r)}{\phi(r)^{1+\epsilon}} \\ &\leq \frac{1}{\epsilon \phi(r_1)^\epsilon} \end{aligned}$$

■

Fix a  $\mathcal{C}^\infty$  volume form  $\Omega$  on  $M$  (it exists since any complex manifold is orientable). There is a well-defined continuous function  $v : M \rightarrow \mathbb{R}^+$  satisfying  $\alpha^m = v\Omega$ . It satisfies the following properties.

**Lemma 2.4.3** *Extend the  $(m, m)$ -form  $\log(v)\alpha^m$  by zero on the vanishing locus of  $v$ . Then  $\log(v)\alpha^m$  is locally integrable on  $M$  and the set*

$$S_\sigma^0 = \{r \geq r_0; \log(v)\eta_r \text{ is not integrable}\}$$

*has measure zero and does not depend on  $\Omega$ .*

Recall that the Ricci curvature form of  $\Omega$  is defined in local coordinates  $(z_1, \dots, z_n)$  such that

$$\Omega = \lambda \prod_{1 \leq j \leq m} \left( \frac{i}{2} dz_j \wedge d\bar{z}_j \right)$$

by the expression

$$\text{Ric } \Omega = -2\pi dd^c \log \lambda.$$

One defines the **Ricci function** by

$$\text{Ric}_\sigma(r) = 2\pi \int_{S(r_0)} \log v \eta_{r_0} - 2\pi \int_{M(r)} \log v \eta_r + T_{\text{Ric } \Omega}(r).$$

This definition is independent of the choice of the volume form  $\Omega$ .

We shall prove the following generalisation of lemma 2.10 in [12]:

**Proposition 2.4.4**

*Consider a pseudo-metric  $h$  defined on the complement  $M \setminus \Sigma$  of a codimension 1 analytic subset  $\Sigma \subset M$  and the associated  $(1, 1)$ -form  $\omega$ . Fix a smooth volume form  $\Omega$  on  $M$ . Let  $\chi$  and  $v$  be the two non-negative smooth functions defined over  $M \setminus \Sigma$  and  $M$  respectively by*

$$\omega^m = \chi \Omega \quad \alpha^m = v \Omega.$$

*Assume that  $\log \chi$  can locally be expressed as the difference of two plurisubharmonic functions. Assume moreover that  $\log \omega^m$  and  $\omega \wedge \alpha^{m-1}$  are locally integrable. Then the Ricci curvature current*

$$\text{Ric}[\omega] = -2\pi dd^c[\log \lambda]$$

*is well-defined and we have*

$$T_{-\text{Ric}[\omega]}(r) \leq_{\text{exc}} -\text{Ric}_\sigma(r) + O(\log T_{[\omega]}(r) + \log r).$$

**Proof:** Denote by

$$\Sigma_h = M \setminus \{z \in M \setminus \Sigma; h_z > 0 \text{ on } T_z M\}$$

the degeneracy locus of  $h$  and  $\tilde{\Sigma} = \Sigma \cup \Sigma_h$ . Let  $\phi = \log \chi$ . By definition, one has on  $M \setminus \tilde{\Sigma}$  the equality of currents:

$$-\text{Ric}[\omega] = 2\pi dd^c[\phi] - \text{Ric } \Omega.$$



Applying Jensen's formula, one obtains:

$$T_{-\text{Ric}[\omega]}(r) = \int_{r_0}^r \frac{dt}{t^{2m-1}} \int_{M(t)} (2\pi dd^c[\phi] - \text{Ric } \Omega) \wedge \alpha^{m-1} \quad (2.8)$$

$$= 2\pi \int_{M(r)} \phi \eta_r + O(1) - T_{\text{Ric } \Omega}(r) \quad (2.9)$$

$$= 2\pi m \int_{M(r)} \log \left( \sigma^{-1} \left( \frac{\chi}{v} \right)^{1/m} \right) \eta_r + 2\pi \int_{M(r)} \log(\sigma^m v) \eta_r - T_{\text{Ric } \Omega}(r) + O(1). \quad (2.10)$$

On the one hand,

$$2\pi \int_{M(r)} \log(\sigma^m v) \eta_r - T_{\text{Ric } \Omega}(r) = -\text{Ric}_\sigma(r) + O(\log r).$$

On the other hand, since the logarithm is concave, one has:

$$\int_{M(r)} \log \left( \sigma^{-1} \left( \frac{\chi}{v} \right)^{1/m} \right) \eta_r \leq \log \left( \int_{M(r)} \sigma^{-1} \left( \frac{\chi}{v} \right)^{1/m} \eta_r \right).$$

A general Fubini's formula claims that for any  $(2m-1)$ -form  $\beta$ ,

$$\int_{M(r)} d\sigma \wedge \beta = \int_{r_0}^r \int_{M(t)} \beta dt$$

hence

$$\frac{d}{dr} \left( \int_{M(r)} d\sigma \wedge \beta \right) = \int_{M(r)} \beta.$$

Using this formula with  $\beta = \sigma^{-1} (\chi/v)^{1/m} d^c \sigma \wedge \alpha^{m-1}$ , we get

$$\begin{aligned} \int_{M(r)} \sigma^{-1} \left( \frac{\chi}{v} \right)^{1/m} \eta_r &= \frac{1}{r^{2m-1}} \int_{M(r)} \sigma^{-1} \left( \frac{\chi}{v} \right)^{1/m} d^c \sigma \wedge \alpha^{m-1} \\ &= \frac{1}{mr^{2m-1}} \frac{d}{dr} \left( \int_{M(r)} \left( \frac{\chi}{v} \right)^{1/m} \alpha^m \right) \text{ by lemma 2.3.1, equation (2.5)} \\ &= \frac{1}{mr^{2m-1}} \frac{d}{dr} \left( \int_{M(r)} \left( \frac{\chi}{v} \right)^{-(m-1)/m} \omega^m \right) \\ &\leq_{\text{exc}} \frac{1}{mr^{2m-1}} \left( \int_{M(r)} \left( \frac{\chi}{v} \right)^{-(m-1)/m} \omega^m \right)^{1+\varepsilon} \end{aligned}$$

according to Borel's lemma

$$\begin{aligned} &\leq_{\text{exc}} \frac{r^{(2m-1)\varepsilon}}{m} \left( \frac{d}{dr} \left( \int_{r_0}^r \frac{dt}{t^{2m-1}} \int_{M(t)} \left( \frac{\chi}{v} \right)^{-(m-1)/m} \omega^m \right) \right)^{1+\varepsilon} \\ &\leq_{\text{exc}} \frac{r^{(2m-1)\varepsilon}}{m} \left( \int_{r_0}^r \frac{dt}{t^{2m-1}} \int_{M(t)} \left( \frac{\chi}{v} \right)^{-(m-1)/m} \omega^m \right)^{(1+\varepsilon)^2} \end{aligned}$$

applying again Borel's lemma.

We would like to compare this last term to  $T_{[\omega]}$ .

Let  $(z_1, \dots, z_m)$  be some local coordinate system. The complex Hessian matrix  $H = \left( \frac{\partial^2 \sigma}{\partial z_i \partial \bar{z}_j} \right)_{i,j}$  is invertible on  $M^+$  and  $\det H = v$ .

Write

$$\omega = \frac{i}{2} \sum_{1 \leq i, j \leq m} a_{ij} dz_i \wedge d\bar{z}_j$$

and denote by  $A$  the matrix  $(a_{ij})_{i,j}$ . One has over  $M^+$ ,

$$\omega^m = \frac{\chi}{v} \alpha^m = \det(AH^{-1}) \alpha^m.$$

Using the concavity of the logarithm,

$$m \det(AH^{-1})^{1/m} \leq \text{Tr}(AH^{-1}).$$

As a consequence,

$$m\omega^m \leq \left( \frac{\chi}{v} \right)^{(m-1)/m} \omega \wedge \alpha^{m-1}. \quad (2.11)$$

Let us come back to the previous inequality. One has

$$\int_{M\langle r \rangle} \sigma^{-1} \left( \frac{\chi}{v} \right)^{1/m} \eta_r \leq_{\text{exc}} \frac{r^{(2m-1)\varepsilon}}{m^{1+(1+\varepsilon)^2}} T_{[\omega]}(r)^{(1+\varepsilon)^2}$$

Finally, (2.8) implies

$$T_{-\text{Ric}[\omega]}(r) \leq_{\text{exc}} 2\pi(1+\varepsilon)^2 \log(T_{[\omega]}(r)) - \text{Ric}_\sigma(r) + O(\log r). \quad (2.12) \quad \blacksquare$$

From proposition 2.4.4 together with lemma 1.1.4, one infers:

### Corollary 2.4.5

In the previous setup, assume that  $\omega$  has at most logarithmic singularities along  $\Sigma$ .  
Then

$$T_{[-\text{Ric}\omega]}(r) \leq_{\text{exc}} 2\pi N_\Sigma(r) - \text{Ric}_\sigma(r) + O(\log T_{[\omega]}(r) + \log r).$$

**Remark 2.4.6** When  $M$  is a parabolic Riemann surface, the Ricci function coincides with the mean Euler characteristic (cf [93], [12])

$$\text{Ric}_\sigma(r) = 2\pi \mathfrak{X}_\sigma(r) := -2\pi \int_{r_0}^r \chi(M(t)) \frac{dt}{t}.$$

To obtain this topological interpretation, one uses a never-vanishing global section of  $K_M^* = TM \cong \mathcal{O}_M$  to construct a smooth global vector field  $V$  having isolated zeroes on  $M$ . The Euler characteristic appears then by applying Poincaré-Hopf index theorem to the vector field  $V$ .

Unfortunately, this machinery cannot be used in the general setting, since the anti-canonical line bundle of  $M$  is not anymore trivial. Actually, one cannot even guarantee in general the existence of a non-trivial global section  $\xi \in H^0(M, \det(TM))$  !

However, if  $K_M$  is indeed trivial, the Ricci function can also be related to the Euler characteristic via the Chern-Gauss-Bonnet formula.

Proposition 2.4.4 extends lemma 2.10 of [12], which we cite below.

**Lemma 2.4.7 (Brotbek-Brunebarbe, [12], Lemma 2.10)** *Let  $(M, \sigma)$  be a non-compact parabolic Riemann surface and  $\xi \in H^0(M, TM)$  a global frame. Let  $\Sigma$  be a reduced divisor on  $M$  and  $h$  be a pseudo-metric on  $M \setminus \Sigma$  such that the function  $\log \|\xi\|_h$  can locally be written as the difference of two subharmonic functions. Assume that the associated  $(1, 1)$ -form  $\omega$  is locally integrable. Then  $\text{Ric}[\omega]$  is well-defined and*

$$T_{-\text{Ric}[\omega]}(r) \leq_{\text{exc}} -2\pi\mathfrak{X}_\sigma(r) + O(\log r + \log T_{[\omega]}(r)).$$

## 2.5 Second main theorem for negative holomorphic sectional curvature

Come back to the previous situation of a parabolic manifold  $(M, \sigma)$  together with a smooth hermitian pseudo-metric  $h$  on  $M \setminus \Sigma$ . Assume that  $h$  has non positive holomorphic bisectional curvature and holomorphic sectional curvature bounded above by a negative constant  $-\gamma < 0$ .

**Lemma 2.5.1** *Denote by  $\omega$  the  $(1, 1)$ -form associated to the pseudo-metric  $h$ . Under these assumptions, one has*

$$[\text{Ric} \omega] \leq -\gamma[\omega]$$

on  $M$ .

**Proof:** Denote by  $H$  the holomorphic bisectional curvature of  $h$ . Let  $x \in M$  and take some local coordinates  $(z_1, \dots, z_m)$  centred at  $x$ . Let  $X \in T_{M,x}$  be a unit holomorphic tangent vector at  $x$  and take an  $h$ -orthonormal basis  $(e_1, \dots, e_m)$  of  $T_{M,x}$  with  $e_1 = X$ .

By definition, one has:

$$\begin{aligned} \text{Ric}(X, X) &= \sum_j H(X, e_j) \\ &= H(e_1, e_1) \text{ since } H \text{ is non positive} \\ &= -\gamma \end{aligned}$$

This completes the proof. ■

**Lemma 2.5.2** *Keep the notations as above. Then:*

1.  $\omega$  is locally integrable on  $M$ .

2. Fix some local coordinate system  $(z_1, \dots, z_n)$  so that

$$\omega^m = i_m \lambda \prod_{1 \leq j \leq m} (dz_j \wedge d\bar{z}_j).$$

Then  $\log \lambda$  can be written as the difference of two plurisubharmonic functions.

3. Fix a smooth volume form  $\Omega$  on  $M$ . It follows that if  $\omega^m = v\Omega$ , then  $\log v$  can locally be written as the difference of two plurisubharmonic functions.

**Proof:** 1. Let  $x$  be a point of  $\Sigma$ . Take a neighbourhood  $W$  of  $x$ . Up to taking a log-resolution of the pair  $(W, W \cap \Sigma)$ , we may assume that  $\Sigma' = W \cap \Sigma$  has simple normal crossings. Take a local holomorphic coordinate system  $(z_1, \dots, z_m)$  centred at  $x$  and such that  $\Sigma' = (z_1 \cdots z_r = 0)$ . We may assume that the  $z_j$  are defined in the set  $\{|z_j| < 1\}$ .

Denote by  $\omega_0$  the following Kähler form on  $W \setminus \Sigma \cong (\Delta^*)^r \times \Delta^{m-r}$ :

$$\omega_0 = \sum_{j=1}^r \frac{1}{|z_j|^2 (\log |z_j|^2)^2} \frac{i}{2\pi} dz_j \wedge d\bar{z}_j + \sum_{j=r+1}^m \frac{i}{2\pi} dz_j \wedge d\bar{z}_j.$$

Since  $\text{Ric } \omega$  is negatively curved, we can apply Schwarz' lemma (see [70], Thm 2.4.14), and there exists a positive constant  $C$  such that

$$\omega \leq C\omega_0. \quad (2.13)$$

Therefore,  $\omega$  is integrable around  $x$ .

2. Using the upper bound 2.13, we see that the function

$$z \mapsto |z_1 \cdots z_r|^2 \lambda(z)$$

is bounded above around  $x$ .

On the other hand, one has

$$2\pi dd^c \log(|z_1 \cdots z_r|^2 \lambda(z)) = -\text{Ric } \omega \geq \gamma \omega \geq 0$$

on  $W \setminus \{x\}$ , so that the function  $\phi : z \mapsto \log(|z_1 \cdots z_r|^2 \lambda(z))$  is plurisubharmonic on  $W \setminus \{x\}$ , therefore extends as a plurisubharmonic function  $\phi$  on the whole of  $W$ .

Hence

$$\log \lambda(z) = \phi(z) - \log(|z_1 \cdots z_r|^2)$$

is itself a plurisubharmonic function. ■

From this result one immediately infers a Second Main Theorem for varieties having a metric with negative holomorphic sectional curvature. Let us first recall a definition. Let  $Y$  be a complex manifold equipped with a smooth Hermitian pseudo-metric  $h$ . Let  $x \in U$

be in the complement of the degeneracy set of  $h$ . The holomorphic sectional curvature of  $h$  at  $x$  in a direction  $\xi \in T_x Y \setminus \{0\}$  is defined to be

$$HSC_{x,\xi}(h) := \sup K_{f^*h}(0)$$

where  $K$  denotes the Gaussian curvature and the supremum is taken over all holomorphic maps  $g : \Delta \rightarrow Y$  such that  $g(0) = x$  and  $\xi \in \mathbb{C} \cdot g'(0)$ . If  $HSC(\omega) \leq -\gamma$  for some  $\gamma > 0$ , we say that the holomorphic sectional curvature is bounded above by  $-\gamma$ . In that case, if  $M$  is a complex manifold and  $f : M \rightarrow Y$  is a holomorphic map such that  $f^*h$  is a pseudo-metric on  $M$ , then its holomorphic sectional curvature is also bounded above by  $-\gamma$ .

### Theorem 2.5.3

Let  $X$  be a smooth projective variety of dimension  $n$  and  $D$  a divisor on  $X$ .

Endow  $X \setminus D$  with a smooth hermitian pseudo-metric  $h$  whose holomorphic sectional curvature is bounded above by a negative constant  $-\gamma$  and holomorphic bisectional curvature is non positive. Assume that the degeneracy set  $\Sigma_h$  of  $h$  is thin.

Let  $(M, \sigma)$  be a parabolic manifold of dimension  $m \leq n$  and  $f : M \rightarrow X$  a non-constant holomorphic map such that  $f(M) \not\subset D$  and  $f(M) \not\subset \Sigma_h$ . Denote by  $\omega$  the  $(1, 1)$ -form associated to the pseudo-metric  $f^*h$  on  $M \setminus f^{-1}(D)_{red}$ . Then  $\omega$  is locally integrable on  $M$  and we have the following Second Main Theorem-type inequality:

$$\gamma T_{[\omega]}(r) \leq_{exc} 2\pi N_{f,D}^{[1]}(r) - \text{Ric}_\sigma(r) + O(\log T_{[\omega]}(r) + \log r).$$

## 2.6 The tautological inequality

### 2.6.1 Introduction

Considering a holomorphic map  $f : M \rightarrow X$ , one wants to estimate the characteristic function of the *tautological line bundle*  $\mathcal{O}(1)$  on the projectivised (logarithmic) cotangent bundle  $\mathbb{P}(\Omega_X^1)$  (or  $\mathbb{P}(\Omega_X(\log D))$ ), using other Nevanlinna functions. This is an example of a so-called *tautological inequality*.

This has been achieved in both the compact and logarithmic cases for  $M = \mathbb{C}$  first (McQuillan [79], Vojta [129]), then  $M = \mathbb{C}^m$  (Gasbarri-Pacienza-Rousseau [54])

In this section, we generalise some of the results of [54] to more general parabolic manifolds. More specifically, we are interested in extending the following.

#### Proposition 2.6.1 (Theorem 3.3 in [54])

Let  $X$  be a smooth projective variety,  $D \subset X$  a simple normal crossings divisor,  $\omega_0$  a Kähler metric on  $X$  and  $f : \mathbb{C}^m \rightarrow X$  a holomorphic mapping of maximal rank such that  $f(\mathbb{C}^m) \not\subset D$ . Set  $X_1 = \mathbb{P}(\wedge^m \Omega_X(\log D))$  and  $L = \mathcal{O}_{X_1}(1)$ . Denote by  $f_1$  the natural meromorphic lift  $\mathbb{C}^m \rightarrow X_1$  of  $f$ . Then

$$T_{f,L}(r) \leq_{exc} N_{f,D}^{[1]}(r) + O(\log T_{f,\omega}(r)).$$

### 2.6.2 The parabolic setting

Our situation is as follows. Let  $X$  be a smooth  $n$ -dimensional projective variety,  $D = D_1 + \dots + D_c$  a simple normal crossing divisor. Let  $A$  be an ample line bundle on  $X$  and  $\omega_A$  the positive first Chern form associated to an appropriate hermitian metric on  $A$ . We consider the projectivisation

$$X_1 = \mathbb{P} \left( \bigwedge^m \Omega_X(\log D) \right)$$

of the vector bundle of logarithmic  $m$ -forms on  $X$ .

Let  $(M, \sigma)$  be a parabolic manifold of dimension  $m$  together with a holomorphic map  $f : M \rightarrow X$  such that  $f(M) \not\subset D$ . Denote  $H = (f^{-1}D)_{red}$ .

Assume moreover that  $m \leq n$  and  $f$  has maximal rank  $m$ , where we define the rank of a holomorphic map to be

$$\text{rk } f = \max\{\text{rk } d_x f; x \in M\}.$$

The subset

$$\Sigma = \{x \in M; \text{rk } d_x f < m\}$$

is analytic of codimension at least 2 (see e.g. [3], lemma 1.14).

The morphism of sheaves

$$f^* \bigwedge^m \Omega_X(\log D) \longrightarrow \bigwedge^m \Omega_M(\log H)$$

induces a rational map

$$f' : M \dashrightarrow \mathbb{P} \left( \bigwedge^m \Omega_X(\log D) \right)$$

lifting  $f$ . The map  $f'$  is well-defined and holomorphic outside an analytic subset  $S$  of codimension at least 2, which is the union of  $\Sigma$  and the locus where  $H$  is not simple normal crossings.

**Nota bene:** In fact, while we only dealt with holomorphic maps, value distribution theory on parabolic manifolds has been developed by Stoll in the more general setting of meromorphic maps. We will not dwell on this subject, but the important point is that all we have done from now on stays valid replacing ‘holomorphic’ by ‘meromorphic’. For more details, the reader is referred to Stoll’s book [121].

The restriction of  $f'$  to  $M \setminus (S \cup H)$  is simply given by differentiating  $f$ . In local coordinates  $t_1, \dots, t_m$ , we have

$$f'|_{M \setminus (S \cup H)}(t_1, \dots, t_m) = \left( f(t_1, \dots, t_m), \left[ \frac{\partial f}{\partial t_1} \wedge \dots \wedge \frac{\partial f}{\partial t_m} \right] \right)$$

We shall prove the following estimation.

**Theorem 2.6.2**

One has

$$T_{f', \mathcal{O}_{X_1}(1)}(r) \leq_{\text{exc}} N_{f,D}^{[1]}(r) - \frac{1}{2\pi} \text{Ric}_\sigma(r) + O(\log T_{f,\omega}(r)).$$

**Proof:** For any  $i = 1, \dots, c$ , let  $s_i$  be a holomorphic section of  $\mathcal{O}_X(D_i)$  defining  $D_i$ , and endow each  $\mathcal{O}_X(D_i)$  with a smooth hermitian metric  $\kappa_i$  such that  $|s_i|_{\kappa_i} < 1$ . Denote  $s = \prod_{i=1}^c s_i$  and  $|s| = \prod_i |s_i|_{\kappa_i}$ .

We can define a smooth metric on  $T_X(-\log D)$  by the singular  $(1, 1)$ -form

$$\omega_1 = \omega_A + \sum_{1 \leq i \leq c} \frac{d|s_i|_{\kappa_i} \wedge d^c |s_i|_{\kappa_i}}{|s_i|_{\kappa_i}^2}.$$

The form  $\omega_1^m$  induces a smooth metric  $\tilde{h}$  on  $\mathcal{O}_{X_1}(-1)$ . Let  $\tilde{\omega}$  be the smooth pseudo-metric  $f^*\omega_1$  on  $M \setminus H$ , with logarithmic singularities along  $H$ . Moreover, the form  $\log(\tilde{\omega}^m)$  is locally integrable along  $H$ : writing  $\omega_1^m = \lambda(z) \prod_{j=1}^n idz_j \wedge d\bar{z}_j$  in some local coordinates  $z_1, \dots, z_n$ , the function  $\log(|s|^2 \lambda)$  is locally bounded and  $\log(|s|^2)$  is plurisubharmonic. Hence  $\log(\lambda) = \log(|s|^2) - \log(|s|^2 \lambda)$  is locally integrable along  $D$ .

By definition, we have

$$-\text{Ric } \tilde{\omega} = 2\pi(f')^* c_1(\mathcal{O}_{X_1}(1), \tilde{h}^{-1}),$$

on  $M \setminus (S \cup H)$ , so that

$$2\pi T_{f', \mathcal{O}_{X_1}(1)} = T_{[-\text{Ric } \tilde{\omega}]}.$$

Define  $\tilde{\omega}_s = \frac{1}{(\log |s \circ f|^2)^2} \tilde{\omega}$ . It is clear that  $\tilde{\omega}_s$  has at most logarithmic singularities along  $H$ . Thus we may apply 2.4.5:

$$T_{[-\text{Ric } \tilde{\omega}_s]} \leq_{\text{exc}} 2\pi N_H(r) - \text{Ric}_\sigma(r) + O(\log T_{[\tilde{\omega}_s]} + \log r).$$

Since  $N_H(r) = N_{f,D}^{[1]}(r)$ , it suffices to prove the following inequalities.

$$T_{[-\text{Ric } \tilde{\omega}]} \leq T_{[-\text{Ric } \tilde{\omega}_s]} + O(\log r + \log T_{f,A}) \quad (2.14)$$

$$\log T_{[\tilde{\omega}_s]} \leq O(\log T_{f,A}) \quad (2.15)$$

Let us first establish (2.14). Over  $M \setminus (H \cup \Sigma)$ , one has

$$-\text{Ric } \tilde{\omega}_s = -\text{Ric } \tilde{\omega} - 2\pi dd^c \log(\log(|s|^2)^{2m}) = -\text{Ric } \tilde{\omega} - 2\pi m dd^c \log(\log(|s|^2)^2). \quad (2.16)$$

Recall that we have chosen the metrics on  $\mathcal{O}(D_i)$  such that  $|s| < 1$ . Hence

$$\log(\log(|s|^2)^2) = 2 \log(-\log(|s|^2)).$$

Because of the moderate singularities of  $\log(-\log(|s|^2))$ , it is readily verified that we can apply Jensen's formula to this function. This gives

$$\begin{aligned}
T_{[dd^c \log(\log(|s|^2)^2)]}(r) &= \int_{M\langle r \rangle} \log(\log(|s|^2)^2) d\mu_r + O(1) \\
&\leq 2 \log \left( \int_{M\langle r \rangle} -\log(|s|^2)^2 d\mu_r \right) + O(1) \\
&\leq 2 \log(-T_{[dd^c \log |s|^2]}(r)) + O(1) \\
&\leq 2 \log \left( -\sum_{i=1}^c N_{f, D_i}(r) + T_{f, \mathcal{O}_X(D)}(r) \right) + O(1) \\
&\leq 2 \log T_{f, \mathcal{O}_X(D)}(r) + O(1) \leq O(\log r + \log T_{f,A})
\end{aligned}$$

Therefore, integrating (2.16) implies (2.14).

By definition,

$$\begin{aligned}
\tilde{\omega}_s &= \frac{1}{(\log |s \circ f|^2)^2} f^* \omega_A + f^* \sum_{i=1}^c \frac{d|s_i|_{\kappa_i} \wedge d^c |s_i|_{\kappa_i}}{|s_i|_{\kappa_i}^2 (\log |s_i|_{\kappa_i}^2)^2} \\
&\leq f^* \omega_A + f^* \sum_{i=1}^c \frac{d|s_i|_{\kappa_i} \wedge d^c |s_i|_{\kappa_i}}{|s_i|_{\kappa_i}^2 (\log |s_i|_{\kappa_i}^2)^2}
\end{aligned}$$

By a straightforward computation, one has

$$\frac{d|s_i|_{\kappa_i} \wedge d^c |s_i|_{\kappa_i}}{|s_i|_{\kappa_i}^2 (\log |s_i|_{\kappa_i}^2)^2} = -\frac{1}{4 \log(|s_i|_{\kappa_i}^2)} dd^c \log(|s_i|_{\kappa_i}^2) - \frac{1}{8} dd^c \log(\log(|s_i|_{\kappa_i}^2)^2).$$

Integrating this relation yields

$$T \left[ f^* \frac{d|s_i|_{\kappa_i} \wedge d^c |s_i|_{\kappa_i}}{|s_i|_{\kappa_i}^2 (\log |s_i|_{\kappa_i}^2)^2} \right] \leq O(T_{f, \mathcal{O}_X(D_i)} + T_{f,A}) \leq O(T_{f,A}),$$

and therefore

$$\log T_{[\tilde{\omega}_s]} \leq O(\log T_{f,A}).$$

■



# 3

## Orbifolds

In what follows, the term *orbifold* will refer to a geometric orbifold pair as defined by Campana. We refer to the papers of Campana [17], Campana-Paun [20] or Campana-Darondeau-Rousseau [19], for instance. The category of geometric orbifolds was introduced to represent ramified coverings of complex manifolds as fully geometric objects. In particular, one wants to define on a geometric orbifold pair the classical invariants for a variety: morphisms, meromorphic maps, differential forms, fundamental groups, universal covers, fields of definition, rational points... One also wants to consider orbifold versions of questions in hyperbolicity. There are indeed appropriate definitions for infinitesimal pseudo-metrics and entire curves in the orbifold context. We will see how geometric orbifold pairs can be considered as an interpolation between compact manifolds and such manifolds endowed with a logarithmic structure. We will indeed denote a geometric orbifold as a pair  $(X, \Delta)$ , where  $X$  is a compact Kähler manifold and  $\Delta$  a  $\mathbb{Q}$ -divisor whose decomposition into irreducible components has coefficients in  $[0, 1]$ .

### 3.1 Campana's orbifold category

#### Definition 3.1.1

A geometric orbifold is a pair  $(X, \Delta)$ , where

- $X$  is a normal complex analytic space;
- $\Delta = \sum_i \left(1 - \frac{1}{m_i}\right) \Delta_i$  is a Weil divisor with coefficients in  $\mathbb{Q} \cap [0, 1]$ .

The rationals  $m_i = a_i/b_i \in \mathbb{Q}_{\geq 1} \cup \{\infty\}$  are the multiplicities of the orbifold divisor. They shall be seen as ramification multiplicities.

If  $X$  is a smooth manifold and  $[\Delta]$  is a divisor with only normal crossings, the orbifold  $(X, \Delta)$  is said smooth.

**Remark 3.1.2** When the  $m_i$  are all infinite (resp. equal to 1), we recover the classical logarithmic (resp. compact) case. One can then naturally regard orbifolds as an interpolation between the compact and the logarithmic cases.

**Remark 3.1.3** When  $X$  is smooth projective and  $\Delta$  has normal crossings support (the situation we will mainly consider), the geometric orbifold  $(X, \Delta)$  is a special case of a smooth Deligne-Mumford stack with coarse moduli space  $X$ .

From now on, we only consider orbifolds  $(X, \Delta)$  whose underlying space  $X$  is a (smooth) complex manifold.

We will now introduce two notions of morphisms from the unit disk to an orbifold.

**Definition 3.1.4**

Let  $(X, \Delta)$  be an orbifold with  $\Delta = \sum_i \left(1 - \frac{1}{m_i}\right) D_i$ ,  $\mathbb{D}$  the unit disk and  $h : \mathbb{D} \rightarrow X$  a holomorphic map. Denote by  $|\Delta|$  the purely logarithmic part of  $\Delta$ , that is the sum of components of  $\Delta$  with infinite multiplicity. Assume that  $h(\mathbb{D}) \cap \text{Supp}[\Delta] = \emptyset$  and  $h(\mathbb{D}) \not\subset |\Delta|$ .

- $h$  is a (non-classical) orbifold morphism from  $(\mathbb{D}, \emptyset)$  to  $(X, \Delta)$  if for any  $i$  and any  $z \in h^{-1}(D_i)$ , the multiplicities satisfy  $\text{mult}_z(h^*D_i) \geq m_i$ . In other words,  $f$  ramifies over each irreducible component  $D_i$  with multiplicity at least  $m_i$ .
- $h$  is a classical orbifold morphism from  $(\mathbb{D}, \emptyset)$  to  $(X, \Delta)$  if for any  $i$  and any  $z \in h^{-1}(D_i)$ , the multiplicity  $\text{mult}_z(h^*D_i)$  is a multiple of  $m_i$ .

More generally, we define morphisms in the orbifold category.

**Definition 3.1.5**

Let  $(X, \Delta_X)$  and  $(Y, \Delta_Y)$  be two orbifolds, and  $f : Y \rightarrow X$  a holomorphic map. We denote by  $m_X, m_Y$  the associated orbifold multiplicities. One says that  $f$  is an (non-classical) orbifold morphism (resp. classical orbifold morphism) when:

1.  $f(Y) \not\subset |\Delta_X|$ ;
2. Let  $D \subset X$ ,  $E \subset Y$  be  $\mathbb{Q}$ -divisors. Take  $t \in \mathbb{Q}$  such that  $f^*D = tE + R$ , for some effective  $\mathbb{Q}$ -divisor  $R$  on  $Y$  not containing  $E$ . Then  $tm_Y(E) \geq m_X(D)$  (resp.  $tm_Y(E)$  is a multiple of  $m_X(D)$ ).

**Remark 3.1.6** 1. The third condition in 3.1.5 is exactly the one preserving sheaves of orbifold differential forms/jet differentials, as well as the morphisms from the unit disk into an integral orbifold.

2. The more restrictive notion of classical orbifold morphisms is necessary to study, for instance, the orbifold fundamental group (for integral orbifolds only). Classical morphisms are also the ones compatible with composition of fibrations.
3. If  $m_X(D) = 1$ , then condition 2 is empty since  $t \in \mathbb{N}_{\geq 1}$  and  $m_Y(E) \geq 1$ . The only conditions to be satisfied are for  $D \subset |\Delta_X|$  and  $f(E) \subset |\Delta_X|$ . There are finitely many when  $X$  and  $Y$  are compact.
4. The composition of two (classical or standard) morphisms  $f : (Y, \Delta_Y) \rightarrow (X, \Delta_X)$  and  $g : (Z, \Delta_Z) \rightarrow (Y, \Delta_Y)$  is still a morphism, provided that  $g \circ f(Z) \not\subset \Delta_X$ .

5. If  $X$  and  $Y$  are compact ( $\Delta_X = 0$ ,  $\Delta_Y = 0$ ), any holomorphic map is an orbifold morphism.
6. If  $(X, \Delta_X)$  is logarithmic,  $f : (Y, \Delta_Y) \rightarrow (X, \Delta_X)$  is an orbifold morphism if and only if  $\Delta_Y \geq f^{-1}(\Delta_X)$ .

## 3.2 Sheaves of orbifold symmetric differential forms

In analogy with the logarithmic setting, we would like to define *orbifold differential forms* as meromorphic differential forms over  $X$  having singularities of order “at most  $1 - 1/m_i$  along each  $D_i$ ”; formally, in some adapted local coordinates, the bundle  $\Omega_{(X, \Delta)}^1$  of orbifold 1-forms should be generated over  $\mathcal{O}_X$  by forms  $\frac{dz_i}{z_i^{1-1/m_i}}$ . To make sense of these objects, one has to work through suitable ramified covers turning  $\Delta$  into an integral divisor.

### 3.2.1 Adapted coverings

Let  $(X, \Delta)$  be a smooth orbifold. Write  $\Delta = \sum_{i \in I} (1 - 1/m_i) \Delta_i$  with either  $m_i = a_i/b_i$  for  $a_i > b_i \geq 0$  coprime, or  $m_i = \infty$ .

#### Definition 3.2.1

A Galois covering  $\pi : Y \rightarrow X$  from a smooth projective variety  $Y$  to  $X$  is said adapted to the pair  $(X, \Delta)$  if

1. for any component  $D_i$ , we have  $\pi^* \Delta_i = p_i D_i$ , where  $p_i$  is a multiple of  $a_i$  and  $D_i$  is a simple normal crossing divisor;
2. both the supports of  $\pi^* \Delta + \text{Ram}(\pi)$  and the branch locus of  $\pi$  have at most normal crossings.

Suppose that  $m_i = a_i \in \mathbb{N}_{\geq 1} \cup \{\infty\}$  for all  $i$ . Then  $\pi$  is said strictly adapted to  $(X, \Delta)$  if  $p_i = m_i$  for all  $i$ .

For a smooth orbifold pair with integer multiplicities, the following proposition ensures that there always exists a strictly adapted covering.

#### Proposition 3.2.2 ([74], Prop. 4.1.12)

Let  $(X, \Delta)$  be a non-singular quasi-projective orbifold such that the support of

$$\Delta = \sum_{i=1}^r (1 - 1/m_i) D_i$$

has only simple normal crossings. Then there exists a smooth variety  $Y$  and a finite flat covering  $h : Y \rightarrow X$  with the property that

$$h^* D_i = m_i D'_i$$

for some smooth divisor  $D'_i$ , for each  $i$  such that  $m_i$  is finite, and  $D' = \sum_{i=1}^r D'_i$  has simple normal crossings.

**Remark 3.2.3** *In general, we cannot expect to find a covering ramifying exactly over  $\text{Supp } \Delta$ .*

We denote for later use, for any  $k \in \mathbb{N} \cup \{\infty\}$ ,

$$\Delta^{(k)} := \sum_{i \in I} (1 - k/m_i)^+ \Delta_i$$

where  $x^+ = \max\{x, 0\}$ . Let  $\pi : Y \rightarrow X$  be an adapted covering. For any point  $y$  in  $Y$ , there exist a neighbourhood  $U \ni y$  invariant under the isotropy group of  $\pi$  with local coordinates  $w_1, \dots, w_n$ , and local coordinates  $z_1, \dots, z_n$  centred at  $\pi(y)$  in  $\pi(U)$  such that the restriction of  $\pi$  to  $U$  has the expression  $\pi(w_1, \dots, w_n) = (w_1^{p_1}, \dots, w_n^{p_n})$ .

The orbifold cotangent bundle associated with  $\pi$  is then defined on  $Y$  as the locally free subsheaf  $\Omega(\pi, \Delta)$  of  $\Omega(\pi, [\Delta])$  fitting in the short exact sequence

$$0 \rightarrow \Omega(\pi, \Delta) \hookrightarrow \pi^* \Omega_X(\log[\Delta]) \rightarrow \bigoplus_{i, m_i < \infty} \mathcal{O}_{\pi^* \Delta_i / m_i} \rightarrow 0.$$

Here the quotient is the composition of the pullback of the residue map

$$\pi^* \text{res} : \pi^* \Omega_X(\log |\Delta|) \rightarrow \bigoplus_{i \in I : m_i < \infty} \mathcal{O}_{\pi^* \Delta_i}$$

with the quotients

$$\mathcal{O}_{\pi^* \Delta_i} \twoheadrightarrow \mathcal{O}_{\frac{\pi^* \Delta_i}{m_i}} = \mathcal{O}_{b_i p_i D_i}.$$

It is locally generated in coordinates as above by the elements

$$w_i^{\frac{p_i}{m_i}} \pi^*(dz_i/z_i) = w_i^{-p_i(1-\frac{1}{m_i})} \pi^*(dz_i). \quad (3.1)$$

We now define a sheaf of orbifold differential forms on  $X$  which is independent of the covering.

**Definition 3.2.4**

*Given an adapted covering  $\pi : Y \rightarrow (X, \Delta)$ , the sheaf of orbifold symmetric differential forms is the direct image sheaf*

$$S^{[N]} \Omega_{(X, \Delta)} = \pi_* \left( (S^N \Omega(\pi, \Delta))^{\text{Aut}(\pi)} \right) \subseteq S^N \Omega_X(\log |\Delta|).$$

*Concretely, in local coordinates as above, it is generated by the elements*

$$\prod_i^n z_i^{[\alpha_i/m_i]} (dz_i/z_i)^{\alpha_i}$$

*such that*

$$\sum_{i=1}^n \alpha_i = N.$$

We see in view of the local expression that this sheaf is indeed independent of the choice of  $\pi$ . Note that in general, the inclusion  $S^{[qr]}\Omega_{(X,\Delta)} \subset S^q(S^{[r]}\Omega_{(X,\Delta)})$  is strict. In fact, we have equality if and only if  $\Delta$  is purely logarithmic.

We would like to extend the classical positivity notions for vector bundles (ample, nef, big, etc).

**Definition 3.2.5**

*The orbifold pair  $(X, \Delta)$  has a big cotangent bundle if  $\Omega(\pi, \Delta)$  is big for some (hence for all) adapted cover  $\pi$ . Equivalently, the orbifold cotangent bundle of  $(X, \Delta)$  is big if for some/any ample line bundle  $A$  on  $X$ , there exists an integer  $N$  such that*

$$H^0(X, S^{[N]}\Omega_{(X,\Delta)} \otimes A^{-1}) \neq \{0\}.$$

To deal with ampleness, we use augmented base loci. The *augmented base locus* of  $\Omega_{(X,\Delta)}$  is defined by

$$\mathbb{B}_+(\Omega_{(X,\Delta)}) = \bigcap_{N \geq 1} \bigcap_{p/q \in \mathbb{Q}} \text{Bs}(S^{[Nq]}\Omega_{(X,\Delta)} \otimes A^{-Np})$$

for some ample line bundle  $A$  over  $X$  (beware that the  $S^{[Nq]}\Omega_{(X,\Delta)}$  involved in the definition are not actual symmetric powers).

Away from  $[\Delta]$ , this set turns out to be independent of the covering  $\pi$ .

**Proposition 3.2.6** ( [31] )

*Over  $X \setminus |\Delta|$ , the image of the augmented base locus  $\mathbb{B}_+(\mathcal{O}_{Y'}(1))$  by the natural projection coincides with the orbifold augmented base locus  $\mathbb{B}_+(\Omega_{(X,\Delta)})$ .*

**Theorem 3.2.7 (Fundamental vanishing theorem)**

*Let  $(X, \Delta)$  be a smooth orbifold with  $X$  projective. Fix an ample line bundle  $A$  on  $X$  and a global orbifold symmetric differential*

$$\omega \in H^0(X, S^{[N]}\Omega_{(X,\Delta)} \otimes A^{-1}).$$

*Then for any orbifold entire curve  $f : \mathbb{C} \rightarrow (X, \Delta)$ ,*

$$f^*\omega \equiv 0.$$

## 3.3 Orbifold hyperbolicity

### 3.3.1 Orbifold Kobayashi pseudo-distance

We recall the following definition for the Kobayashi pseudo-distance on a smooth manifold  $X$ : it is the largest pseudo-distance  $d_X : X \times X \rightarrow \mathbb{R}_+$  such that  $d_X \leq h^*(d_{\mathbb{D}})$ , for any holomorphic map  $h : \mathbb{D} \rightarrow X$ ,  $d_{\mathbb{D}}$  being the Poincaré distance on the unit disk  $\mathbb{D}$ . We can copy this definition and adapt it directly to the orbifold setting as below.

**Definition 3.3.1**

*The orbifold Kobayashi pseudo-distance  $d_{(X,\Delta)}$  on an orbifold  $(X, \Delta)$  is the largest pseudo-distance on  $X \setminus \text{Supp}[\Delta]$  such that any orbifold morphism  $h : (\mathbb{D}, \emptyset) \rightarrow (X, \Delta)$  is distance-decreasing with respect to the Poincaré distance on the unit disk, i.e.*

$$d_{(X,\Delta)} \leq h^*d_{\mathbb{D}}.$$

**Remark 3.3.2** *If one considers only classical morphisms  $h: \mathbb{D} \rightarrow (X, \Delta)$ , one obtains the classical Kobayashi pseudo-distance  $d_{(X, \Delta)}^*$ , with clearly  $d_{(X, \Delta)} \leq d_{(X, \Delta)}^*$ . Over  $X \setminus |\Delta|$ , one has*

$$d_X \leq d_{(X, \Delta)} \leq d_{(X, \Delta)}^* \leq d_{X \setminus |\Delta|}.$$

As in the usual setting, there is an equivalent definition for the Kobayashi pseudo-distance using chains of holomorphic disks.

**Proposition 3.3.3**

*For any  $z, z' \in X \setminus \text{Supp}[\Delta]$ , the Kobayashi pseudo-distance  $d_{(X, \Delta)}$  is the infimum of the sums*

$$\sum_i d_{\mathbb{D}}(p_i, q_i)$$

*over all finite chains  $f_1, \dots, f_r$  of orbifold morphisms from  $\mathbb{D}$  to  $(X, \Delta)$  such that  $f_1(p_1) = z$ ,  $f_i(q_i) = f_{i+1}(p_i)$ ,  $f_r(q_r) = z'$ .*

An immediate consequence of the definition is the distance-decreasing property of orbifold morphisms.

**Proposition 3.3.4**

*Let  $h: (X, \Delta) \rightarrow (X', \Delta')$  is an orbifold morphism (resp. classical orbifold morphism). Then*

$$h^* d_{(X', \Delta')} \leq d_{(X, \Delta)}$$

*(resp.  $h^* d_{(X', \Delta')}^* \leq d_{(X, \Delta)}^*$ ).*

**Definition 3.3.5**

*The orbifold  $(X, \Delta)$  is hyperbolic (resp. classically hyperbolic) if the pseudo-distance  $d_{(X, \Delta)}$  (resp.  $d_{(X, \Delta)}^*$ ) is non-degenerate.*

### 3.3.2 An orbifold Brody theorem

A corollary of 3.3.4 is

**Corollary 3.3.6** *Let  $(X, \Delta)$  be a hyperbolic orbifold. Then every orbifold morphism  $f: (\mathbb{C}, \emptyset) \rightarrow (X, \Delta)$  is constant.*

An orbifold entire curve inside  $(X, \Delta)$  will naturally be a non-constant orbifold morphism  $f: \mathbb{C} \rightarrow (X, \Delta)$ .

In [21], we can find the following generalisation of Brody's reparametrisation lemma to orbifolds.

**Theorem 3.3.7**

*Let  $(X, \Delta)$  be a compact orbifold. Assume that  $(X, \Delta)$  is not hyperbolic, i.e.  $d_{(X, \Delta)}$  is not a distance. Then there exists a non-constant holomorphic map  $f: \mathbb{C} \rightarrow X$  which either is an orbifold morphism or satisfies  $f(\mathbb{C}) \subset |\Delta|$ . Furthermore,*

$$\sup \|f'(z)\| = \|f'(0)\| > 0.$$

**Proposition 3.3.8** ( [31] )

Let  $(X, \Delta)$  be a smooth orbifold pair, with  $\Delta = \sum_i (1 - 1/m_i) \Delta_i$ . Assume that a sequence of orbifold maps  $h_n : D \rightarrow (X, \Delta)$  from the unit disk to  $(X, \Delta)$  converges locally uniformly to a holomorphic map  $h : D \rightarrow X$ . Let

$$X_h = \bigcap_{h(D) \subset \Delta_i} \Delta_i, \Delta_h = \bigcap_{h(D) \not\subset \Delta_i} \Delta_i.$$

Then  $h$  is an orbifold map  $D \rightarrow (X_h, \Delta_h)$ .

As an immediate consequence, reasoning exactly as in [21], we obtain the following result.

**Theorem 3.3.9**

Consider a smooth orbifold pair  $(X, \Delta)$  as above. For a subset of indices  $I$ , let  $\Delta_I = \bigcap_{i \in I} \Delta_i$ , and let

$$\Delta(I^c) = \sum_{j \notin I} (1 - 1/m_j) \Delta_j.$$

If all pairs  $(\Delta_I, \Delta(I^c)|_{\Delta_I})$  are Brody-hyperbolic, then the pair  $(X, \Delta)$  is Kobayashi-hyperbolic.

**3.3.3 Conjectures**

It is natural to generalise hyperbolicity conjectures on projective varieties to the orbifold case. Namely, one expects

**Conjecture 3.3.10 (Green-Griffiths, orbifold version)** Let  $(X, \Delta)$  be a smooth projective orbifold of general type, i.e. such that  $K_X + \Delta$  is a big  $\mathbb{Q}$ -divisor. Then there exists a proper subvariety  $Z \subsetneq X$  containing every non-constant orbifold entire curve  $f : \mathbb{C} \rightarrow (X, \Delta)$ .

Sporadic results have been obtained so far in the direction of this conjecture, in particular in dimension 1 and 2 ( [99], [100], [19], [18] ).

This conjecture has been proved by Brotbek and Deng in [14] for a smooth boundary divisor of sufficiently large degree.

**Theorem 3.3.11 (Brotbek-Deng '19 [14])**

Let  $X$  be a smooth projective variety of dimension  $n \geq 2$ . Let  $D \subset X$  be a general smooth hypersurface of degree  $d$  with

$$d \geq (n + 2)^{n+3} (n + 1)^{n+3}.$$

Then the orbifold  $(X, (1 - 1/d)D)$  is Kobayashi-hyperbolic.

Campana, Darondeau and Rousseau treated the case of an arrangement of 11 lines in  $X = \mathbb{P}^2$ .

**Theorem 3.3.12** ( [19] )

On  $X = \mathbb{P}^2$ , let  $\Delta$  consist of 11 lines in general position with orbifold multiplicity 2. Then the orbifold  $(X, \Delta)$  is Brody-hyperbolic, so that the orbifold Green-Griffiths-Lang conjecture holds.

The case of in smooth orbifold surfaces has recently been studied by Rousseau, Turchet and Wang. They obtained algebraic degeneracy of orbifold curves under numerical conditions on the intersection of the irreducible components of the divisor.

**Theorem 3.3.13 (Rousseau-Turchet-Wang '21 [101])**

Let  $X$  be a smooth projective surface and  $\Delta = \sum_{i=1}^q (1 - 1/m_i)D_i$  with  $q \geq 2$  such that the support of  $\Delta$  has normal crossings. Assume that there exist positive integers  $p_1, \dots, p_q$  such that

- the divisor  $D_p := p_1 D_1 + \dots + p_q D_q$  is ample;
- for every  $i = 1, \dots, q$ , the following inequality holds:

$$2D_p^2 \xi_i > (D_p \cdot D_i) \xi_i^2 + 3D_p^2 p_i,$$

where  $\xi_i$  is the minimal positive solution of the equation  $D_i^2 x^2 - 2(D_p \cdot D_i)x + D_p^2 = 0$ .

Then for large enough multiplicities  $m_i$ ,

1. every orbifold entire curve  $f : \mathbb{C} \rightarrow (X, \Delta)$  is algebraically degenerate;
2. the orbifold  $(X, \Delta)$  is pseudo algebraically hyperbolic.

In a recent preprint, Guo and Wang showed algebraic degeneracy for orbifold curves inside  $(\mathbb{P}^2, \Delta)$  when  $\Delta$  has 3 components.

**Theorem 3.3.14 ([66])**

Let  $F_1, F_2, F_3$ , be homogeneous irreducible non-constant polynomials in  $\mathbb{C}[X_0, X_1, X_2]$ . Assume that the plane curves  $D_i = \{F_i = 0\} \subset \mathbb{P}^2$ ,  $1 \leq i \leq 3$  intersect transversely. Consider an orbifold divisor  $\Delta = \sum_{i=1}^3 (1 - 1/m_i)D_i$ . Suppose that

$$\deg \Delta = \sum_{i=1}^3 \frac{m_i - 1}{m_i} \deg P_i > 3.$$

Then there exists a positive real  $\ell$  and a positive integer  $N$  such that if  $\min\{m_1, m_2, m_3\} \leq \ell$  then the image of any orbifold entire curve  $f : \mathbb{C} \rightarrow (\mathbb{P}^2, \Delta)$  is contained in a plane curve of degree at most  $N$ .



# 4

## Hyperbolicity of quasi-projective varieties

The content of this chapter makes up the preprint [39].

### 4.1 Introduction

The classical Little Picard Theorem can be reinterpreted in the context of complex hyperbolicity, by saying that the complement of three points in  $\mathbb{P}^1$  is Brody-hyperbolic. Generalising Picard's result, a theorem of Green provides examples in higher dimension.

#### Theorem 4.1.1

*The complement of  $c \geq 2n + 1$  general hyperplanes in  $\mathbb{P}^n$  is hyperbolic and hyperbolically embedded.*

Green [58] actually proved a more general fact.

#### Theorem 4.1.2 (Green '72)

*Let  $f : \mathbb{C} \rightarrow \mathbb{P}^n$  be an entire curve taking values in the complement of  $n + k \geq n + 1$  hyperplanes in general position. Then  $f(\mathbb{C})$  is contained in a linear subspace of dimension  $\lfloor n/k \rfloor$ .*

We obtain then that entire curves in the complement of  $n + 2$  general hyperplanes are algebraically degenerate.

This theorem is illustrated by the following statement, due to Bloch and Cartan.

#### Theorem 4.1.3

*Let  $H_0, \dots, H_{n+1}$  be  $n + 2$  hyperplanes in general position in  $\mathbb{P}^n$ . If  $f : \mathbb{C} \rightarrow \mathbb{P}^n \setminus \bigcup_i H_i$  is a non-constant holomorphic map, then its image lies in a diagonal hyperplane.*

In the statement, a diagonal hyperplane is defined as the hyperplane generated by

$$\bigcap_{i \in I} H_i \cong \mathbb{P}^{n-|I|} \quad \text{and} \quad \bigcap_{i \in I^c} H_i \cong \mathbb{P}^{|I|-1}$$

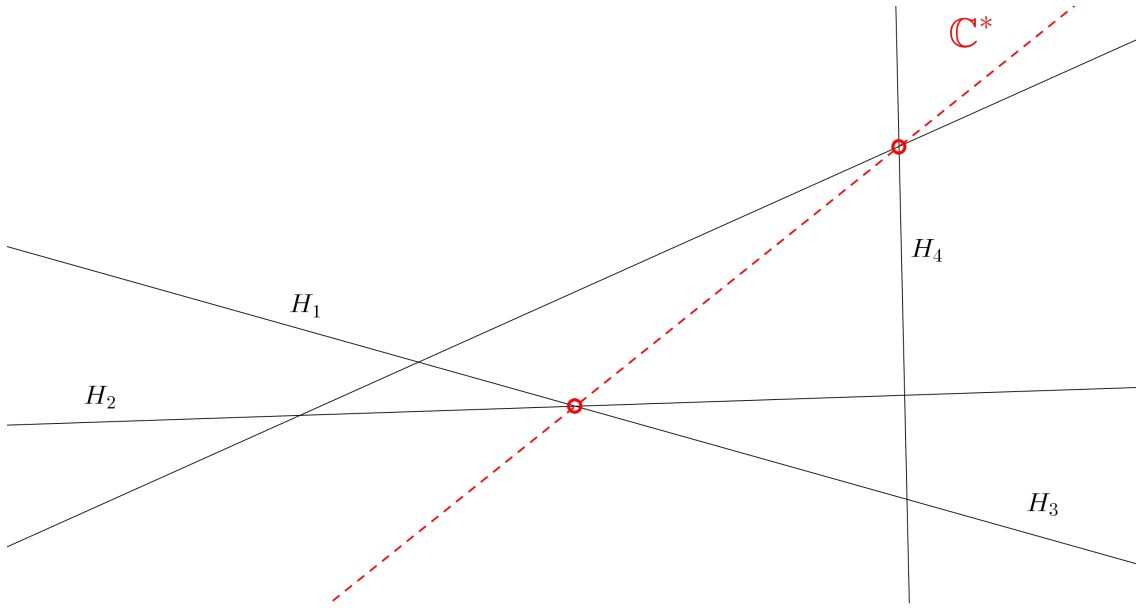


Figure 4.1: The complement of four lines in  $\mathbb{P}^2$  is not hyperbolic

for some subset of indices  $I \subset \{0, \dots, n\}$ . The case  $n = 2$  is represented on figure 4.1 below.

More generally, let us consider a smooth log pair  $(X, D)$ . It has been shown by several authors that hyperbolicity properties of  $X \setminus D$  could be investigated through the positivity of the associated logarithmic cotangent bundle  $\Omega_X(\log D)$ , see e.g. [86], [13], [14], [31], [19], [18].

When studying the properties of the logarithmic cotangent bundle, one finds trivial quotients supported on the different components of  $D$ . Each such quotient gives a positive dimensional subvariety of the augmented base locus  $\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_X(\log D))}(1))$ . In particular,  $\Omega_X(\log D)$  cannot be ample in any way. A natural question to ask then is whether these quotients are the only obstructions to ampleness.

There are now many examples for which the logarithmic cotangent bundle is big. However, if one asks for a stronger positivity property, there are up to now very few results. The first one in this direction is due to Noguchi. It concerns general arrangements of lines in  $\mathbb{P}^2$ . It can be formulated with our vocabulary as follows.

**Proposition 4.1.4 (Noguchi '86 [86])**

*The logarithmic tangent bundle along an arrangement  $D$  of  $c \geq 6$  lines in general position with respect to hyperplanes and quadrics is ample modulo  $D$ .*

This has been generalised to higher dimension in [31].

**Theorem 4.1.5 (Darondeau-Rousseau '20, [31] Theorem A)**

*The logarithmic cotangent bundle along an arrangement  $\mathcal{A}$  of  $c \geq \binom{n+2}{2}$  hyperplanes in  $\mathbb{P}^n$  in general position with respect to hyperplanes and to quadrics is ample modulo  $\mathcal{A}$ .*

On the other hand, Brotbek-Deng [13] proved that the logarithmic cotangent bundle associated to a smooth pair  $(X, D)$  is almost ample when the degree of the irreducible components of  $D$  is very large.

**Theorem 4.1.6 (Brotbek-Deng '18 [13])**

Let  $X$  be a smooth projective variety of dimension  $n$  and let  $c \geq n$ . Let  $L$  be a very ample line bundle on  $X$ . For any  $m \geq (4n)^{n+2}$  and for general hypersurfaces  $H_1, \dots, H_c \in |L^m|$ , writing  $D = \sum_i H_i$ , the logarithmic cotangent bundle  $\Omega_X(\log D)$  is almost ample.

More generally, we can ask similar questions in the orbifold case, which interpolates between the compact and the logarithmic cases. Following Campana (see [16]), a smooth orbifold is a pair  $(X, \Delta)$  where  $X$  is a smooth complex projective variety and  $\Delta$  is a  $\mathbb{Q}$ -divisor with normal crossings support and coefficients in  $\mathbb{Q} \cap [0, 1]$ . One can define a sheaf  $\Omega^1(X, \Delta)$  of differential forms with ‘fractional’ poles: these are actually holomorphic forms living on a suitable ramified covering of  $X$ .

Theorem 4.1.1 already provides an explicit condition on a hyperplane arrangement to have hyperbolic and hyperbolically embedded complement. The aim of this chapter is to investigate further the properties of the attached logarithmic cotangent bundle and the geometry of its augmented base locus. Among other results, we compute the dimension of this augmented base locus in terms of the number of hyperplanes in the arrangement. We derive similar results in the orbifold setting. We also get applications concerning hyperbolicity of Fermat varieties.

## 4.2 Some projective geometry

In this section, we discuss topics of classical algebraic geometry that will be used thereafter.

### 4.2.1 Varieties of linear spaces

Let  $\mathbb{G}(k, n) = \text{Grass}(k+1, n+1)$  be the Grassmannian of  $k$ -linear subspaces in  $\mathbb{P}^n$ . This is a subvariety of  $\mathbb{P}(\wedge^{k+1} \mathbb{C}^{n+1})$  of dimension  $(k+1)(n-k)$ . We define a subvariety  $\Sigma \subset \mathbb{G}(k, n) \times \mathbb{P}^n$  by setting

$$\Sigma = \{(Z, p); p \in Z\}.$$

Let  $\pi_1, \pi_2$  be the projection maps from  $\Sigma$  to  $\mathbb{G}(k, n)$  and  $\mathbb{P}^n$  respectively. We can make use of  $\Sigma$  to construct subvarieties of  $\mathbb{P}^n$  swept out by  $k$ -planes.

**Proposition 4.2.1**

Let  $\Phi \subset \mathbb{G}(k, n)$  be any subvariety. Then

$$X = \bigcup_{Z \in \Phi} Z = \pi_2(\pi_1^{-1}(\Phi))$$

is a subvariety of  $\mathbb{P}^n$ .

**Rational normal scrolls**

A particular case of this construction is obtained when  $\Phi \cong \mathbb{P}^1$ . Let  $k \geq 2$ . A subvariety  $X \subset \mathbb{P}^n$  given as the union

$$X = \bigcup_{\lambda \in \mathbb{P}^1} Z_\lambda$$

of a 1-parameter family of  $(k - 1)$ -planes is named a *rational normal  $k$ -fold scroll*.

One can show ([67], Theorem 19.9) that  $X$  has minimal degree  $1 + n - k$ .

In particular, a hypersurface scroll  $X$  is a quadric. According to [67], Ex 8.36,  $X$  can also be characterised equivalently as:

- a quadric of rank 3 or 4, in the sense that the symmetric matrix representing  $X$  has rank 3 or 4;
- a cone over a plane conic or a smooth quadric surface in  $\mathbb{P}^3$ ;
- an irreducible quadric containing an  $(n - 2)$ -plane.

### 4.2.2 Dual varieties

Consider the  $n$ -dimensional projective space  $\mathbb{P}^n$ . The *dual projective space*  $(\mathbb{P}^n)^*$  is defined as the projective space parameterising hyperplanes in  $\mathbb{P}^n$ .

Let  $X$  be a subvariety of  $\mathbb{P}^n$  of dimension  $k$ . We define the dual variety  $X^* \subset (\mathbb{P}^n)^*$  to be the closure of the set of all hyperplanes tangent to  $X$  at a smooth point.

Restricting first to the case where  $X$  is a smooth hypersurface, we may also say that  $X^*$  is the locus of hyperplanes  $H \subset \mathbb{P}^n$  such that the intersection  $H \cap X$  is singular. It is indeed a variety since it is the image of  $X$  under the *Gauss map*  $\mathcal{G}_X$ , a morphism sending a point  $p \in X$  to its embedded tangent hyperplane  $\mathbb{T}_p X$ ; in coordinates, if  $f$  is a homogeneous polynomial defining  $X$ , the Gauss morphism is given by

$$\mathcal{G}_X : p \mapsto \left[ \frac{\partial F}{\partial Z_0}(p) : \cdots : \frac{\partial F}{\partial Z_n}(p) \right].$$

If  $X \subset \mathbb{P}^n$  is a *smooth* hypersurface of degree  $d \geq 2$ , the morphism  $\mathcal{G}_X$  is birational onto its image as well as finite. The dual  $X^*$  is then again a hypersurface, though in general highly singular (never smooth if  $d \geq 3$ ). One can compute its degree  $\deg X^* = d(d - 1)^{n-1}$ . In particular, the dual of a smooth quadric hypersurface is again a smooth quadric.

In the general situation, we define the *conormal variety* as the closure  $CX$  of the incidence correspondence

$$\{(p, H) \in \mathbb{P}^n \times (\mathbb{P}^n)^* \mid p \in X_{sm} \text{ and } \mathbb{T}_p X \subset H\}.$$

The dual variety  $X^*$  is then naturally defined as the image  $\pi_2(CX) \subset \mathbb{P}^{n*}$  of the second projection.

A fundamental result concerning dual varieties is the Reflexivity theorem.

#### Theorem 4.2.2

*Let  $X \subset \mathbb{P}^n$  be an irreducible variety and  $X^* \subset \mathbb{P}^{n*}$  its dual. Then the conormal variety  $CX \subset \mathbb{P}^n \times \mathbb{P}^{n*}$  is equal to  $CX^* \subset \mathbb{P}^{n*} \times \mathbb{P}^n \cong \mathbb{P}^n \times \mathbb{P}^{n*}$ . It follows that  $(X^*)^* = X$ : the dual of a variety is the variety itself.*

Considering a (possibly singular) subvariety  $X \subset \mathbb{P}^n$ , what can the dimension of  $X^*$  be? One computes easily the dimension of  $CX$ : the fibre of the first projection

map  $CX \rightarrow X$  over a smooth point  $p \in X_{sm}$  is simply the subspace  $\mathbb{P}^{n-k-1} \subset \mathbb{P}^{n*}$  of hyperplanes containing the  $k$ -plane  $\mathbb{T}_p X$ . Hence the inverse image in  $CX$  of the smooth locus of  $X$  is irreducible of dimension  $n-1$ . We conclude that  $X^*$  is irreducible of dimension less than  $n-1$ , and that it should be  $(n-1)$ -dimensional in most cases. More precisely,  $X^*$  has dimension  $n-1$  exactly when the general tangent hyperplane to  $X$  is tangent at only finitely many points.

### Example 4.2.3

*A situation where  $X^*$  fails to be a hypersurface is when the tangent hyperplanes are constant along subvarieties of  $X$ : for example, when we have a variety ruled by linear subspaces, as cones over subvarieties of  $\mathbb{P}^n$ . Typical examples are given by rational normal scrolls of dimension  $k \geq 3$ . Recall that a  $k$ -dimensional scroll  $X = \bigcup_{\lambda \in \mathbb{P}^1} Z_\lambda$  is a variety ruled in  $\mathbb{P}^{k-1}$ 's. The tangent plane  $\mathbb{T}_p X$  at any point  $p \in X$  will naturally contain the  $(k-1)$ -plane  $Z_\lambda$  of the ruling through  $p$ . We conclude that the general fibre of the second projection  $CX \rightarrow \mathbb{P}^{n*}$  is  $(k-2)$ -dimensional, so that  $X^*$  has dimension  $n-k+1$ .*

*For instance, if  $X$  is a hypersurface scroll in  $\mathbb{P}^n$ , or in other words a quadric hypersurface of rank  $r \leq 4$ , then its dual  $X^*$  is a surface inside a  $\mathbb{P}^3$  in the non-degenerate case ( $r=4$ ) and a plane conic curve otherwise.*

## 4.3 Positivity of the logarithmic cotangent bundle

### 4.3.1 Positivity of logarithmic pairs

Let  $(X, D)$  be a smooth log pair, that is,  $X$  is a smooth projective variety and  $D$  is a simple normal crossings divisor. The logarithmic tangent bundle  $T_X(-\log D)$  is defined as the locally free subsheaf of  $T_X$  of vector fields tangent to  $D$ . If  $(z_1, \dots, z_n)$  are local coordinates on an open set  $U \subset X$  such that  $U \cap D = (z_1 \dots z_r = 0)$ , then  $T_X(-\log D)|_U$  is generated by

$$z_1 \frac{\partial}{\partial z_1}, \dots, z_r \frac{\partial}{\partial z_r}, \frac{\partial}{\partial z_{r+1}}, \dots, \frac{\partial}{\partial z_n}.$$

Its dual is the  $\mathcal{O}_X$ -module of meromorphic 1-forms with at most logarithmic poles, locally generated by the forms

$$\frac{dz_1}{z_1}, \dots, \frac{dz_r}{z_r}, dz_{r+1}, \dots, dz_n.$$

It is called the logarithmic cotangent bundle of the pair and denoted  $\Omega_X(\log D)$ .

Write  $D = \sum_{1 \leq i \leq c} D_i$  the decomposition of  $D$  into irreducible components, and  $D(I) = \sum_{i \in I} D_i$  for  $I \subset \{1, \dots, c\}$ . Then the logarithmic cotangent bundle fits in the residue exact sequence

$$0 \longrightarrow \Omega_X \longrightarrow \Omega_X(\log D) \xrightarrow{\text{Res}} \bigoplus_i \mathcal{O}_{D_i} \longrightarrow 0.$$

Here the residue morphism is given over any open subset  $U \subset X$  by

$$\text{Res} \left( \alpha + \sum_{i=1}^c \beta_i \frac{d\sigma_i}{\sigma_i} \right) = (\beta_i|_{D_i})_{1 \leq i \leq c},$$

where  $\sigma_i$  is a section defining  $D_i$  over  $U$ .

We have several other residue exact sequences corresponding to the various subsets of components of  $D$

$$0 \longrightarrow \Omega_X(\log D(I^{\mathbb{L}})) \longrightarrow \Omega_X(\log D) \xrightarrow{\text{Res}} \bigoplus_{i \in I} \mathcal{O}_{D_i} \longrightarrow 0.$$

Its restriction to  $D_I$  leads to a trivial quotient

$$\Omega_X(\log D)|_{D_I} \xrightarrow{\text{Res}} \mathcal{O}_{D_I}^{\oplus |I|}$$

which induces an inclusion

$$\tilde{D}_I := \mathbb{P}(\mathcal{O}_{D_I}^{\oplus r}) \cong D_I \times \mathbb{P}^{r-1} \hookrightarrow \mathbb{P}(\Omega_X(\log D)).$$

Hence we see that the logarithmic cotangent bundle is never ample if  $D$  is not empty.

The projectivisation of these quotients are unavoidable obstructions to the ampleness of the tautological bundle in the projectivised cotangent bundle, see [13]. A natural question to ask then is whether these are the only ones.

**Definition 4.3.1**

We say that the pair  $(X, D)$  has **almost ample logarithmic cotangent bundle** if

$$\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_X(\log D))}(1)) \subset \bigcup_{|I| \leq n-1} \tilde{D}_I.$$

We will also consider the following slightly weaker notion.

**Definition 4.3.2**

Let  $p : \mathbb{P}(\Omega_X(\log D)) \rightarrow X$  be the natural projection. The logarithmic cotangent bundle is said **ample modulo boundary** if  $p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_X(\log D))}(1))) = D$ .

These properties are of great importance in hyperbolicity, because of the vanishing theorem below.

**Theorem 4.3.3 (Fundamental vanishing theorem, logarithmic version [84])**

Let  $f : \mathbb{C} \rightarrow X \setminus D$  be an entire curve. For any ample line bundle  $A$  on  $X$  and any global logarithmic symmetric form  $\omega \in H^0(X, S^m \Omega_X(\log D) \otimes A^{-1})$ ,

$$f^* \omega \equiv 0.$$

In other words, for any entire curve  $f : \mathbb{C} \rightarrow X \setminus D$ , one has

$$f(\mathbb{C}) \subset p\left(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_X(\log D))}(1))\right).$$

Denoting by  $f' : \mathbb{C} \rightarrow \mathbb{P}(\Omega_X(\log D))$  the natural lift of  $f$ , one even has

$$f'(\mathbb{C}) \subset \mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_X(\log D))}(1)).$$

Hence, if  $\Omega_X(\log D)$  is ample modulo  $D$ , then the complement  $X \setminus D$  is Brody-hyperbolic.

### 4.3.2 The map $\Phi$

Let  $L \rightarrow X$  be a holomorphic line bundle on a smooth projective variety  $X$ . The evaluation map

$$H^0(X, L) \otimes \mathcal{O}_X \longrightarrow L$$

induces a rational map

$$\Phi_L : X \dashrightarrow \mathbb{P}H^0(X, L).$$

By definition,  $L$  is ample (resp. big) if and only if  $\Phi_{L^m}$  is an embedding (resp. is birational onto its image) for some  $m > 0$ . If  $L$  is globally generated,  $\Phi_L$  is a morphism, which we will denote  $\Phi$  if there is no ambiguity.

We have the following characterisation of the augmented base locus.

#### Proposition 4.3.4

*With the above notation, suppose that  $L$  is globally generated. Then the augmented base locus  $\mathbb{B}_+(L)$  is the union of all positive-dimensional fibres of  $\Phi_L$ , i.e.*

$$\mathbb{B}_+(L) = \{w \in X; \dim \Phi_L^{-1}(\Phi_L(w)) > 0\}.$$

**Proof:** Since  $L$  is globally generated,  $\Phi_L$  is a well-defined morphism on  $X$  and  $L = \Phi_L^* \mathcal{O}_{\mathbb{P}H^0(X, L)}(1)$ . So we can apply Lemma 4.3.5 below with  $A = \mathcal{O}_{\mathbb{P}H^0(X, L)}(1)$ . ■

**Lemma 4.3.5** *Let  $Y$  be a projective variety and consider a regular morphism  $f : X \rightarrow Y$ . For any ample line bundle  $A$  on  $Y$ , the augmented base locus of  $f^*A$  is the union of the positive-dimensional fibres of  $f$ , i.e.*

$$\mathbb{B}_+(f^*A) = \{x \in X; \dim_x f^{-1}(\{f(x)\}) > 0\} =: \text{NFL}(f).$$

**Proof:** By Nakamaye's Theorem 1.2.16,

$$\mathbb{B}_+(f^*A) = \bigcup_{\substack{Z \subset X \\ Z \cdot f^*A^{\dim Z} = 0}} Z.$$

If  $Z \subset X$  is not contained in  $\text{NFL}(f)$ , then the restriction  $f|_Z$  is generically finite. But this implies that

$$Z \cdot f^*A^{\dim Z} = (\deg f|_Z) f(Z) \cdot A^{\dim Z} > 0.$$

The reverse inclusion is straightforward since by definition

$$f(\text{NFL}(f)) \cdot A^{\dim \text{NFL}(f)} = 0. \quad \blacksquare$$

### 4.3.3 The case of hyperplane arrangements

Let us now study the special case of hyperplane arrangements in  $\mathbb{P}^n$ . We start with two observations.

**Lemma 4.3.6 (Brotbek-Deng '18 [13], Prop. 4.1)** *Let  $D = \sum_{i=1}^c D_i$  be a simple normal crossings divisor in  $\mathbb{P}^n$  with  $c < n$ . Then for any  $m > 0$ ,*

$$H^0(\mathbb{P}^n, S^m \Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(-1)) = 0.$$

*In particular,  $\Omega_{\mathbb{P}^n}(\log D)$  is not almost ample.*

**Lemma 4.3.7 ([13], Prop. 4.2)** *If  $D = \sum_{i=1}^c D_i$  is an arrangement of  $c$  hypersurfaces in general position in  $\mathbb{P}^n$ , then the associated logarithmic cotangent bundle satisfies  $h^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(\log D)) = c - 1$ .*

**Proof:** For  $1 \leq i \leq c$ , let  $s_i \in H^0(\mathbb{P}^n, \mathcal{O}(\lambda_i))$  be a section defining the hypersurfaces  $D_i$ . We claim that the forms  $\omega_i = \lambda_1 d \log s_i - \lambda_i d \log s_1$  form a basis of global sections of  $\Omega_{\mathbb{P}^n}(\log D)$ . First, these are indeed well-defined global sections. If we denote respectively by  $x_1, \dots, x_n$  and  $y_1, \dots, y_n$  the standard affine coordinates over the open subsets  $U_0 = \{Z_0 \neq 0\}$  and  $U_1 = \{Z_1 \neq 0\}$ , we have over  $U_0 \cap U_1$

$$\begin{aligned} \lambda_1 d \log s_i(x) - \lambda_i d \log s_1(x) &= \lambda_1 d \log(x_1^{\lambda_i} s_i(y)) - \lambda_i d \log(x_1^{\lambda_1} s_1(y)) \\ &= \lambda_1 d \log s_i(y) - \lambda_i d \log s_1(y). \end{aligned}$$

The residue exact sequence induces a long exact sequence in cohomology

$$0 \longrightarrow H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(\log D)) \xrightarrow{\text{Res}} \bigoplus_{i=1}^c H^0(D_i, \mathcal{O}_{D_i}) \cong \mathbb{C}^c \xrightarrow{\delta} H^1(\mathbb{P}^n, \Omega_{\mathbb{P}^n}).$$

Denoting by  $e_i \in H^0(D_i, \mathcal{O}_{D_i})$  the constant section equal to 1, we compute  $\text{Res}(\omega_i) = \lambda_1 e_i - \lambda_i e_1$ . As a consequence, we get  $\lambda_1 \delta(e_i) = \lambda_i \delta(e_1)$ . In particular, the map  $\delta$  has rank 1, so that the dimension of  $H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(\log D))$  is exactly  $c - 1$ .  $\blacksquare$

**Lemma 4.3.8** *The logarithmic cotangent bundle associated with an arrangement of  $c \geq n + 1$  hyperplanes  $H_j$  in general position in  $\mathbb{P}^n$  is globally generated.*

**Proof:** Without loss of generality, we can assume that  $H_j = (Z_j = 0)$  for  $j = 0, \dots, n$ .

The forms  $\omega_j = \frac{dZ_j}{Z_j} - \frac{dZ_0}{Z_0}$  are global sections of  $\Omega_{\mathbb{P}^n}(\log D)$ . Hence a point  $z$  in the base locus of  $\Omega_{\mathbb{P}^n}(\log D)$  would satisfy  $Z_j \xi_0 - Z_0 \xi_j = 0$  for every  $\xi \in \mathbb{P}^n$ , which is impossible.  $\blacksquare$

From now on, we fix an arrangement  $D = \sum_{i=0}^{n+k} H_i$  of  $n + k + 1 \geq n + 2$  hyperplanes in general position in  $\mathbb{P}^n$ . Without loss of generality, we assume that the first  $n + 1$  components are the coordinate hyperplanes  $H_i = (\ell_i(Z) = Z_i = 0)$ . For  $j = 1, \dots, k$ , let  $\ell_{n+j} = a_0^j Z_0 + \dots + a_n^j Z_n$  be a linear form defining the hyperplane  $H_{n+j}$ .



In view of Proposition 4.3.4, in order to study the augmented base locus of  $\Omega_{\mathbb{P}^n}(\log D)$ , we have to compute the fibres of the map

$$\Phi := \Phi_{\mathcal{O}_{\mathbb{P}^n}(\log D)}(1).$$

To do so, we describe explicitly the morphism  $\Phi$  local in coordinates.

Let us work on the affine chart  $U_0 = \{Z_0 \neq 0\} \subset \mathbb{P}^n$  with standard coordinates  $z_1, \dots, z_n$ . By the proof of Lemma 4.3.7, a basis of global sections of  $\Omega_{\mathbb{P}^n}(\log D)$  is given by

$$\frac{dz_1}{z_1}, \dots, \frac{dz_n}{z_n}, \frac{d\ell_{n+1}}{\ell_{n+1}}, \dots, \frac{d\ell_{n+k}}{\ell_{n+k}}.$$

On the open set  $U = U_0 \cap \bigcap_{j=1, \dots, k} (\ell_{n+j} \neq 0)$ , a frame for  $T_{\mathbb{P}^n}(-\log D)$  is given by  $z_1 \frac{\partial}{\partial z_1}, \dots, z_n \frac{\partial}{\partial z_n}$ . Therefore, we may identify  $\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D)|_U)$  with  $U \times \mathbb{P}^{n-1}$  via

$$\begin{aligned} U \times \mathbb{P}^{n-1} &\longleftrightarrow \mathbb{P}(\Omega_{\mathbb{P}^n}(\log D)|_U) \\ (z, [\xi]) &\longmapsto \left( z, \left[ \xi_1 z_1 \frac{\partial}{\partial z_1} + \dots + \xi_n z_n \frac{\partial}{\partial z_n} \right] \right) \end{aligned}$$

Since  $\Omega_{\mathbb{P}^n}(\log D)$  is globally generated with  $h^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(\log D)) = n+k$ , we have a quotient

$$H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(\log D)) \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow \Omega_{\mathbb{P}^n}(\log D) \rightarrow 0$$

inducing a map

$$\Phi : \mathbb{P}(\Omega_{\mathbb{P}^n}(\log D)) \rightarrow \mathbb{P}(H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(\log D)) \otimes \mathcal{O}_{\mathbb{P}^n}) \simeq \mathbb{P}^n \times \mathbb{P}^{n+k-1} \rightarrow \mathbb{P}^{n+k-1}.$$

The morphism  $\Phi$  restricted to  $U$  then has the following expression:

$$\begin{aligned} \Phi(z, [\xi_1 : \dots : \xi_n]) &= \left[ \frac{dz_1}{z_1} \cdot \xi : \dots : \frac{dz_n}{z_n} \cdot \xi : \frac{d\ell_{n+1}}{\ell_{n+1}} \cdot \xi : \dots : \frac{d\ell_{n+k}}{\ell_{n+k}} \cdot \xi \right] \\ &= \left[ \xi_1 : \dots : \xi_n : \frac{\sum_{j=1}^n a_j^1 \xi_j z_j}{a_0^1 + \sum_{j=1}^n a_j^1 z_j} : \dots : \frac{\sum_{j=1}^n a_j^k \xi_j z_j}{a_0^k + \sum_{j=1}^n a_j^k z_j} \right]. \end{aligned}$$

From this we can obtain an explicit description of the fibres of  $\Phi|_U$ . Take a vector  $V = [V_1 : \dots : V_n : W_1 : \dots : W_k] \in \mathbb{P}^{n+k-1}$ . The fibre of  $\Phi|_U$  above  $V$  is described by

$$\Phi(z, [\xi]) = V \iff \begin{cases} \xi_i = \lambda V_i, & i = 1, \dots, n \\ \sum_{j=0}^n a_j^l (V_j - W_l) z_j = 0, & l = 1, \dots, k \end{cases} \text{ for some } \lambda \in \mathbb{C} \setminus \{0\}.$$

Therefore

$$p(\Phi^{-1}(V)) \cap U = \left\{ (z_1, \dots, z_n); \sum_{j=0}^n a_j^l (V_j - W_l) z_j = 0 \text{ for all } l = 1, \dots, k \right\},$$

which is a linear subspace of  $\mathbb{P}^n$  (to simplify notation, we let here  $z_0 = 1$  and  $\xi_0 = V_0 = 0$ ).

Note that we can work out the same proof for any open subset  $\mathbb{P}^n \setminus \bigcup_{i \notin I} H_i$ ,  $|I| = n$ , up to a linear coordinate change. Since linear subspaces keep the same shape after a linear transformation, the result holds over the whole of  $\mathbb{P}^n$ .

Together with the results of 4.3.2, we have therefore proven that:

**Proposition 4.3.9**

The logarithmic cotangent bundle  $\Omega_{\mathbb{P}^n}(\log D)$  fails to be ample modulo  $D$  if and only if the projection of the augmented base locus  $\mathbb{B}_+(\mathcal{O}_{\mathbb{P}^n}(\log D)(1))$  contains a line not in  $\text{Supp } D$ .

In other words,  $\Omega_{\mathbb{P}^n}(\log D)$  is ample modulo  $D$  if and only if its restriction to any line  $l \not\subset \text{Supp } D$  is ample.

**4.3.4 Construction of obstruction lines**

In this section, we will use some vocabulary and results of [47]. We introduce some notation. Denote by  $(\mathbb{P}^n)^*$  the dual projective  $n$ -space, that is the projective space of hyperplanes in  $\mathbb{P}^n$ . For a linear subspace  $Z$  of  $\mathbb{P}^n$ , we denote by  $Z^*$  the dual of  $Z$ , i.e. the projective space of hyperplanes containing  $Z$ . This is a subspace of  $(\mathbb{P}^n)^*$  of dimension  $\text{codim}(Z) - 1$ .

As in subsection 4.3.3, we consider a logarithmic pair  $(\mathbb{P}^n, D)$ , where  $D = H_0 + \dots + H_{n+k}$  is an arrangement of  $n + k + 1 \geq n + 2$  hyperplanes in general position. Remark that we can already assume that we have at least  $2n + 1$  hyperplanes:  $\mathbb{P}^n \setminus D$  is indeed Kobayashi-hyperbolic if  $\Omega_{\mathbb{P}^n}(\log D)$  is ample modulo  $D$  (see e.g. [33]), and it is well-known that the complement of  $2n$  hyperplanes in  $\mathbb{P}^n$  is never hyperbolic. We assume without loss of generality that the  $n + 1$  first components are the coordinate hyperplanes, i.e.  $H_i = (Z_i = 0)$  for  $i = 0, \dots, n$ . We aim to prove the following result.

**Theorem 4.3.10**

The logarithmic cotangent bundle of the pair  $(\mathbb{P}^n, D)$  **fails to be ample modulo  $D$**  if, and only if, there exists a line  $l \not\subset D$  such that  $l^* \cong \mathbb{P}^{n-2}$  and the points  $H_i^*$  are contained in a same quadric hypersurface in  $\mathbb{P}^{n*}$ .

Let us first explain how this condition is translated in the original space  $\mathbb{P}^n$ . Let  $X \subset \mathbb{P}^{n*}$  denote a quadric hypersurface containing an  $(n-2)$ -plane. According to [67],  $X$  is a quadric of rank  $\leq 4$ , swept out by a 1-parameter family of  $(n-2)$ -planes:  $X = \bigcup_{\lambda \in \mathbb{P}^1} Z_\lambda$ . We have seen in 4.2.3 that the dual variety  $X^*$  is in general a non-degenerate quadric surface contained in a  $\mathbb{P}^3 \subset \mathbb{P}^n$  (when  $X$  has rank  $\leq 3$ ,  $X$  is a cone over a plane conic, and  $X^*$  is also a cone over the dual conic curve). It is actually a ruled surface  $\mathbb{P}^1 \times \mathbb{P}^1$ , whose two rulings correspond dually to the two rulings in  $\mathbb{P}^{n-2}$ 's of  $X$ .

**Remark 4.3.11** *Consequently, the condition that the hyperplanes  $H_i$  belong to a same quadric rational scroll  $X$  in  $\mathbb{P}^{n*}$  means that the  $H_i$  are all tangent to the dual quadric surface  $X^* \subset \mathbb{P}^n$ .*

For simplicity, considering  $n + k + 1$  hyperplanes  $H_0, \dots, H_{n+k}$  in general position in  $\mathbb{P}^n$ , we introduce the following property. Theorem 4.3.10 asserts that this is equivalent to  $\Omega_{\mathbb{P}^n}(\log D)$  being ample modulo  $D$ .

**Condition** ★ *The points  $H_0^*, \dots, H_{n+k}^*$  in the dual projective space  $\mathbb{P}^{n*}$  corresponding to the hyperplanes  $H_0, \dots, H_{n+k}$  do not belong to a same quadric hypersurface of rank at most 4.*

### Corollary 4.3.12

*The logarithmic cotangent bundle of the pair  $(\mathbb{P}^n, D)$  is ample modulo  $D$  if, and only if, the hyperplanes  $H_i$  impose at least  $4n - 2$  linearly independent conditions on quadrics. In particular, if  $D$  has no more than  $4n - 3$  components, there always exist obstructions to ampleness away from  $D$ .*

Note that for  $n = 2$ , we retrieve Noguchi's result with six lines on  $\mathbb{P}^n$ . For  $n = 3$ , we obtain the same bound as in [31], but as soon as  $n \geq 4$ , our result is strictly better.

**Proof:** Following the previous section, the strategy of the proof relies on the search for lines contracted by the map  $\Phi$ . Since the logarithmic cotangent bundle is globally generated,  $\Phi$  is a morphism and an embedding when restricted to each fibre of the projection  $\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D)) \rightarrow \mathbb{P}^n$ .

According to Proposition 4.3.15 below, there exists a line  $l \not\subset \text{Supp } D$  contracted by  $\Phi$  if, and only if, there exists a degree 1 rational map  $l \rightarrow H_0$  preserving each component  $H_i$ . Applying further 4.3.16, producing such a map amounts to the fact that the points of  $(\mathbb{P}^n)^*$  corresponding to the hyperplanes  $H_i$  lie on a same quadric  $X$  of rank  $\leq 4$ .

Moreover, we can compute the dimension of the space of quadrics of rank  $\leq 4$  on  $\mathbb{P}^n$ . Choosing such a form amounts to taking a 4-codimensional subspace of  $\mathbb{C}^{n+1}$  together with a quadratic form on  $\mathbb{C}^4$ . We find that the dimension equals  $4(n - 3) + \frac{4 \times 5}{2} - 1 = 4n - 3$ .

This ends the proof of both the theorem and its corollary. ■

We shall now prove in two main steps why the existence of such obstruction lines away from  $\text{Supp } D$  is equivalent to the above condition concerning quadrics.

Let us first give some terminology.

### Definition 4.3.13 ( [47] )

*A line  $l \subset \mathbb{P}^n$  is called superjumping for the pair  $(\mathbb{P}^n, D)$  if the restriction  $\Omega_{\mathbb{P}^n}(\log D)|_l$  is not ample.*

Using the residue exact sequence, it is clear that all the lines belonging to  $\text{Supp } D$  are superjumping.

In other words, Proposition 4.3.9 says that the logarithmic cotangent bundle  $\Omega_{\mathbb{P}^n}(\log D)$  is ample modulo boundary if and only if there is no superjumping line except those in  $\text{Supp } D$ . The following characterisation of superjumping lines is straightforward.

### Lemma 4.3.14

*A line  $l \subset \mathbb{P}^n$  is superjumping if, and only if, the restriction  $T_{\mathbb{P}^n}(-\log D)|_l$  of the logarithmic tangent bundle has a non-vanishing global section.*

**Proof:** Recall that a vector bundle on  $\mathbb{P}^1$  can be decomposed as a direct sum of line bundles. Since  $\Omega_{\mathbb{P}^n}(\log D)|_l$  is globally generated, there exist positive integers  $a_1, \dots, a_r$  such that

$$\Omega_{\mathbb{P}^n}(\log D)|_l \cong \mathcal{O}_l(a_1) \oplus \cdots \oplus \mathcal{O}_l(a_r) \oplus \mathcal{O}_l^{\oplus n-r},$$

so that

$$T_{\mathbb{P}^n}(-\log D)|_l \cong \mathcal{O}_l(-a_1) \oplus \cdots \oplus \mathcal{O}_l(-a_r) \oplus \mathcal{O}_l^{\oplus n-r}.$$

By definition, the line  $l$  is superjumping if  $r < n$ , *i.e.* if there is a trivial factor in the decomposition of  $T_{\mathbb{P}^n}(-\log D)|_l$ . ■

The next proposition is the key argument in the proof of the theorem. Note that even though this result appears as Prop. 7.5 in [47], our proof is completely different and self-contained. Ours is geometric and constructs explicitly the map whereas the authors of [47] use algebraic constructions called *elementary transformations*, in order to reconstruct inductively the logarithmic cotangent bundle, adding each component of  $D$  one by one.

### Proposition 4.3.15

Let  $l$  be a line not lying in any hyperplane  $H_i$ . Denote by  $p_i$  the point corresponding to  $H_i$  in the dual projective space  $(\mathbb{P}^n)^*$ , and by  $Z$  the 2-codimensional linear subspace of  $(\mathbb{P}^n)^*$  corresponding to  $l$ . Then the following are equivalent:

1. there exists a non-zero (even non-vanishing) global section of the vector bundle  $T_{\mathbb{P}^n}(-\log D)|_l$  (*i.e.*,  $l$  is superjumping);
2. there exists a regular map  $\psi : l \rightarrow H_0$  of degree 1 such that for all  $1 \leq i \leq n+k$ ,  $\psi(l \cap H_i) \subset H_0 \cap H_i$ .

**Proof:** We are going to construct in coordinates the desired map. Let  $X_0, \dots, X_n$  be homogeneous coordinates on  $\mathbb{P}^n$  such that  $H_0 = (X_0 = 0)$  and  $l = (X_2 = \cdots = X_n = 0)$ . On the open subset  $U_0 = \mathbb{P}^n \setminus H_0$ , consider the inhomogeneous coordinates  $(z_1, \dots, z_n) = (X_1/X_0, \dots, X_n/X_0)$ . Then  $z_1$  defines a coordinate on  $l \setminus H_0 \simeq \mathbb{C}$ .

Assume first that there exists  $\xi \in H^0(l, T_{\mathbb{P}^n}(-\log D)|_l) \setminus \{0\}$ . We can write

$$\xi = \xi_1(z_1) \frac{\partial}{\partial z_1} + \cdots + \xi_n(z_1) \frac{\partial}{\partial z_n}$$

for some holomorphic functions  $\xi_j : \mathbb{C} \rightarrow \mathbb{C}$ .

Define now inhomogeneous coordinates on  $U_1 = \mathbb{P}^n \setminus \{X_1 = 0\}$  by

$$(x_0, x_2, \dots, x_n) = (X_0/X_1, X_2/X_1, \dots, X_n/X_1) = (1/z_1, z_2/z_1, \dots, z_n/z_1).$$

The Jacobian matrix associated to this change of coordinates in  $U_0 \cap U_1$  is given by

$$\begin{pmatrix} -1/z_1^2 & 0 & \cdots & 0 \\ -z_2/z_1^2 & 1/z_1 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ -z_n/z_1^2 & \cdots & \cdots & 1/z_1 \end{pmatrix}.$$

On  $l \cap U_0 \cap U_1$ , the vector field  $\xi$  writes as

$$\xi(1/x_0) = -x_0^2 \xi_1(1/x_0) \frac{\partial}{\partial x_0} + x_0 \xi_2(1/x_0) \frac{\partial}{\partial x_2} + \cdots + x_0 \xi_n(1/x_0) \frac{\partial}{\partial x_n}.$$

Since  $\xi$  needs to be holomorphic, the functions  $x_0^2 \xi_1(1/x_0), x_0 \xi_2(1/x_0), \dots, x_0 \xi_n(1/x_0)$  are holomorphic in the variable  $x_0$ . This shows that the holomorphic functions  $x_j : \mathbb{C} \rightarrow \mathbb{C}$  are in fact polynomials, of degree  $\leq 2$  for  $j = 1$  and  $\leq 1$  otherwise. Moreover,  $\xi \in H^0(l, T_{\mathbb{P}^n}(-\log H_0)|_l)$ , so we can write, for some holomorphic function  $g$ ,

$$-x_0^2 \xi_1 \left( \frac{1}{x_0} \right) = x_0 g(x_0).$$

It follows that  $\xi_1$  also has degree  $\leq 1$ .

Define naturally a map by

$$\begin{aligned} \psi : l &\rightarrow H_0 \\ z &\mapsto [0 : \xi_1(z) : \cdots : \xi_n(z)] \end{aligned}.$$

According to the proof of Lemma 4.3.14, a global section of  $T_{\mathbb{P}^n}(-\log D)$  come from the trivial factors of the decomposition, so vanishes nowhere. Hence  $\psi$  is well-defined. By definition,  $\xi$  is tangent to every component of  $D$ ; this directly implies that  $\psi$  sends each  $H_j$  into itself.

Conversely, assume that we are given a degree 1 map  $\psi : l \rightarrow H_0$  as above. Over  $U_0 = \mathbb{P}^n \setminus H_0$ , write  $\psi = [0 : \xi_1 : \cdots : \xi_n]$ , and choose representatives  $\xi_j$ , which are degree 1 polynomials in the variable  $z_1$ . Then we can define a non-trivial logarithmic vector field by

$$\xi = \sum_{j=1}^n \xi_j(z_1) \frac{\partial}{\partial z_j} \in H^0(l \setminus H_0, T_{\mathbb{P}^n}(-\log H_0)|_l).$$

By construction, one has  $\xi(l \cap H_j) \in T_{l \cap H_j} H_j$  for all  $j$ , hence

$$\xi \in H^0(l \setminus H_0, T_{\mathbb{P}^n}(-\log D)|_l). \quad \blacksquare$$

More concretely, given a global logarithmic vector field  $\xi \in H^0(l, T_{\mathbb{P}^n}(-\log D)|_l)$ , we construct the map  $\psi : l \rightarrow H_0$  in the following way (see figure 4.2 below for an illustration of the 2-dimensional case): starting from a point  $q$  in  $l$ , we have a tangent vector to  $l$  at  $x$ , which uniquely defines a line  $L_x$  through  $x$ . Denote by  $\psi(x)$  the intersection point of

$L_x$  with  $H_0$ . By definition, the sections of  $T_{\mathbb{P}^n}(-\log D)|_l$  are those vector fields tangent to all  $H_i$ 's. Hence, if  $x \in H_i \cap l$ , the point  $\psi(x)$  is also in  $H_i$ .

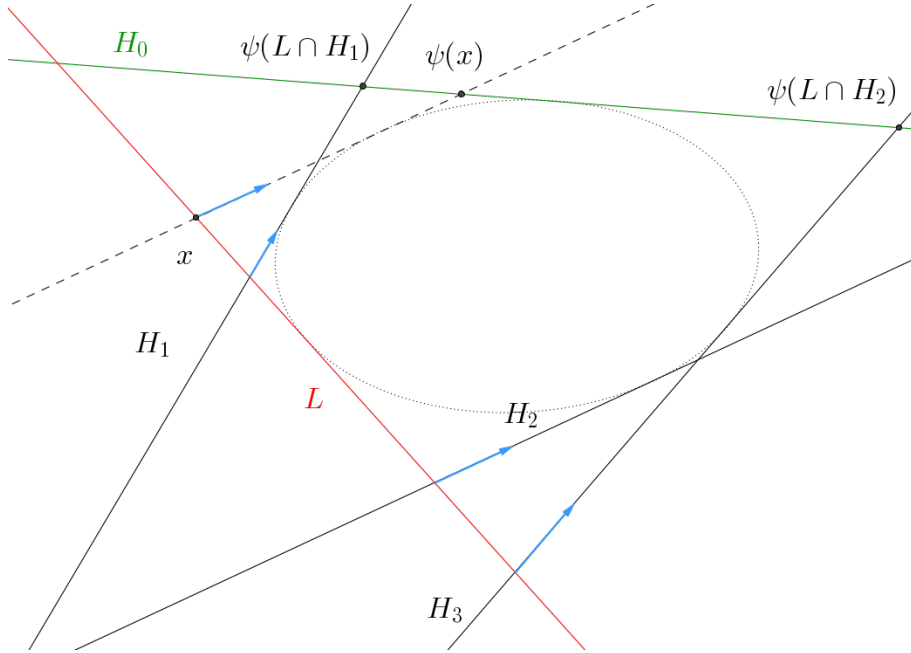


Figure 4.2: Lines in  $\mathbb{P}^2$

Let  $p_0, \dots, p_{n+k}$  be the points of  $(\mathbb{P}^n)^*$  corresponding dually to the hyperplanes  $H_0, \dots, H_{n+k}$  and let  $W = l^* \subset (\mathbb{P}^n)^*$  be the  $(n-2)$ -plane of hyperplanes in  $\mathbb{P}^n$  containing  $l$ . Assume that there exists a quadric hypersurface  $X \subset (\mathbb{P}^n)^*$  containing  $W, p_0, \dots, p_{n+k}$ .

The inclusion  $W \cup \{p_0, \dots, p_c\} \subset X$  is equivalent to saying that all the hyperplanes  $H_i$  are tangent to the dual surface  $X^*$  and that  $l \subset X^*$ .

The next proposition provides a necessary and sufficient condition for the existence of such a quadric hypersurface.

**Proposition 4.3.16**

Consider  $c+1$  points  $p_0, p_1, \dots, p_c$  in general position in  $\mathbb{P}^n$  (i.e., no  $k \leq n+1$  of them belong to a same  $(k-2)$ -plane), and a  $(n-2)$ -plane  $W$  disjoint from  $\{p_0, p_1, \dots, p_c\}$ . For  $i = 1, \dots, c$ , denote by  $L_i = \langle W, p_i \rangle$  the hyperplane generated by  $W$  and  $p_i$ . The following two assertions are equivalent.

1. There exists a quadric hypersurface scroll  $X \subset \mathbb{P}^n$  containing  $W$  and all the points  $p_i$ .
2. There exists a degree 1 map  $\psi : \mathbb{P}^1 \cong W^* \rightarrow \mathbb{P}^{n-1} \cong p_0^*$  such that  $\psi(L_i)$  contains the point  $p_i$  and  $\psi(L_0) \neq L_0$ .

**Proof:** 1. Consider a degree 1 map  $\psi$  as described in the statement. Then  $\psi$  sends  $W^* \cong \mathbb{P}^1$  to a line in  $\mathbb{P}^{n-1} \cong p_0^*$ . The intersection  $W' = \bigcap_{L \in W^*} \psi(L)$  is therefore another  $(n-2)$ -plane.

Define a subset of  $\mathbb{P}^n$  by

$$X(\psi) = \bigcup_{L \in W^*} L \cap \psi(L).$$

It contains  $W$  as well as the points  $p_i$ . Indeed, for any  $q \in W \setminus W'$ , there exists a unique hyperplane  $\psi(L)$  containing both  $W'$  and  $q$ . It follows that  $q \in L \cap \psi(L) \subset X(\psi)$ . Moreover, by definition of  $\psi$ , we know that  $p_i \in L_i \cap \psi(L_i)$  for all  $i$ , so that  $p_i \in X(\psi)$ .

By construction,  $X(\psi)$  is an  $(n-1)$ -dimensional scroll of degree 2 in  $\mathbb{P}^n$ , which contains  $W \cup \{p_0, \dots, p_c\}$ .

2. Assume the existence of the quadric  $X$ . We will show that  $X$  is necessarily of the form  $X(\psi)$  for some map  $\psi$ .

The set  $W^*$  of hyperplanes containing  $W$  is a projective line  $\mathbb{P}^1$ . Let  $L_\infty \in W^*$  be the tangent hyperplane to  $X$  at any point of  $W$ . Since  $p_0 \in X \setminus W$ ,  $p_0 \notin L_\infty$ . For any  $L \in W^*$ , the set  $L \cap X$  is a quadric hypersurface inside  $L$ , which contains  $W$ . Hence there exists an  $(n-2)$ -plane  $W(L)$  such that  $L \cap X = W \cup W(L)$ . One has  $W(L) \neq W$  except for  $L = L_\infty$ . Moreover,  $p_0 \in W(L)$  only if  $L = L_0$ .

We define a map  $\psi : W^* \rightarrow p_0^*$  by setting  $\psi(L) = \langle W(L), p_0 \rangle$  for  $L \neq L_0$ , and  $\psi(L_0) = \mathbb{T}_{p_0}X$  (embedded tangent hyperplane to  $X$  at  $p_0$ ). Note that if since  $p_i \in X \setminus W$ ,  $p_i \in W(L_i)$ . Then  $\psi$  has the desired properties, with  $\psi(L_\infty) = L_0$ .

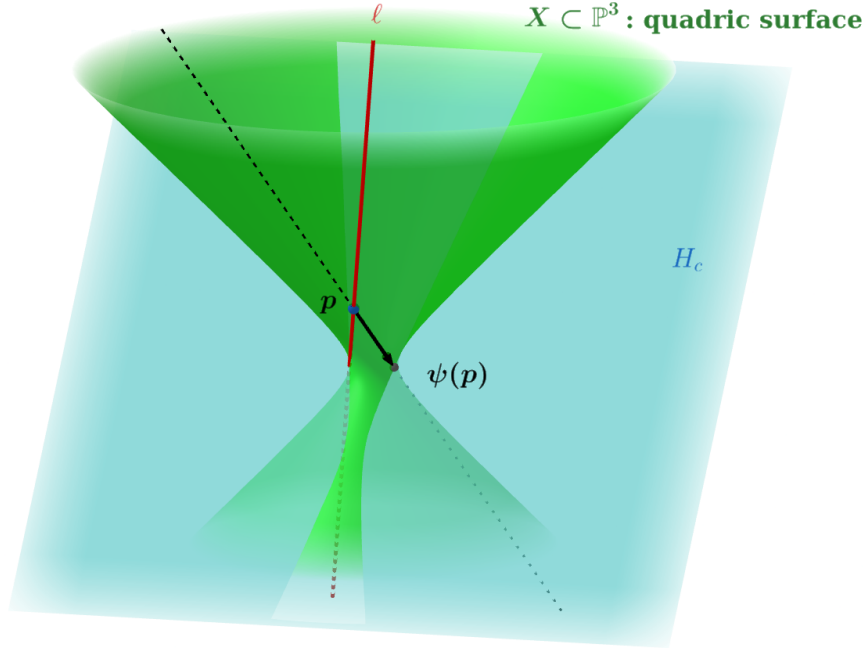
Moreover, for any hyperplane  $L$ , one has  $Z(L) = L \cap \psi(L)$ , and  $X$  is described as the union

$$\bigcup_{L \in W^*} L \cap \psi(L). \quad \blacksquare$$

**Remark 4.3.17** *We can also understand this construction in the original space  $\mathbb{P}^n$ .*

*Consider a ruled quadric surface  $X \cong \mathbb{P}^1 \times \mathbb{P}^1$  containing the line  $l$  and tangent to all hyperplanes  $H_i$ .*

*A point  $p \in l$  belongs to two lines of  $X$ , one in each ruling. Let  $l_p \subset X$  be the other line through  $p$ . The map  $\psi$  sends  $p$  to the intersection point of  $l_p$  with  $H_0$ . This argument is illustrated by figure 4.3 below.*

Figure 4.3: Construction for planes in  $\mathbb{P}^3$ 

### 4.3.5 Almost ampleness

Let  $(X, D)$  be a smooth log pair. Whereas being ample modulo  $D$  only involves the image of  $\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_X(\log D))}(1))$  in  $X$ , the stronger notion of almost ampleness reads on the augmented base locus itself. In general, this last property is strictly stronger. However, in the particular case of hyperplane arrangements, both are equivalent.

#### Theorem 4.3.18

Let  $D = \sum_{i=0}^{n+k} D_i$  be an arrangement of hyperplanes in general position in  $\mathbb{P}^n$ . Then  $\Omega_{\mathbb{P}^n}(\log D)$  is ample modulo  $D$  if, and only if, it is almost ample.

**Proof:** We only need to treat one direction. Assume that the logarithmic cotangent bundle associated to  $D$  is ample modulo  $D$ . Then  $\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_X(\log D))}(1)) \subset p^{-1}(D)$ .

We will study more carefully the fibres of  $\Phi$  above the strata of  $D$ . According to Lemma 4.3.4, their union forms the augmented base locus  $\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_X(\log D))}(1))$ .

We need to show that  $\Phi$  restricted to  $p^{-1}(D) \setminus \bigcup_{I: |I| < n} \tilde{D}_I$  is injective.

Let  $I \subset \{0, \dots, n+k\}$  be a subset of indices of cardinality  $< n$ . Up to a linear coordinate change, we can assume that  $I = \{1, \dots, r\}$  with  $r \leq n$ .

Again, let us work on the open subset  $U = (Z_0 \neq 0) \cap \bigcap_{1 \leq j \leq k} (\ell_{n+j} \neq 0)$  with standard affine coordinates  $(z_1, \dots, z_n)$ . In these coordinates,

$$D_I = \{(0, \dots, 0, z_{r+1}, \dots, z_n)\} \cong \mathbb{P}^{n-r-1}.$$

Moreover, the restricted residue map  $\text{Res} : \Omega_{\mathbb{P}^n}(\log D)|_{D_I \cap U} \rightarrow \mathcal{O}_{D_I}^{\oplus r}$  is given by the



expression

$$\text{Res}|_{D_I \cap U} \left( \sum_{i=1}^n \eta_i \frac{dz_i}{z_i} \right) = (\eta_1, \dots, \eta_r).$$

Under the above trivialisation,  $\tilde{D}_I$  is described as

$$\tilde{D}_I \cap p^{-1}(U) \cong \left\{ (z, [\xi_1 : \dots : \xi_n]) \in U \times \mathbb{P}^{n-1} \left| \begin{array}{l} z_i = 0 \ \forall 1 \leq i \leq r \\ \text{and } \xi_i = 0 \\ \forall i \geq r+1 \end{array} \right. \right\}.$$

Remark first that for any  $J \subset \{0, \dots, n+k\}$ ,  $p^{-1}(D_I) \cap \tilde{D}_J = \emptyset$  unless  $J \subset I$ . This can be easily checked in coordinates.

For all  $(z, [\xi]) \in p^{-1}(D_I)$ , we have

$$\Phi(z, [\xi]) = \left[ \xi_1 : \dots : \xi_n : \frac{\sum_{j=r+1}^n a_j^1 \xi_j z_j}{a_0^1 + \sum_{j=r+1}^n a_j^1 z_j} : \dots : \frac{\sum_{j=r+1}^n a_j^k \xi_j z_j}{a_0^k + \sum_{j=r+1}^n a_j^k z_j} \right]. \quad (\diamond)$$

Note that for  $J \subset I$  and  $(z, [\xi]) \in p^{-1}(D_I) \cap \tilde{D}_J$ , one has

$$\Phi(z, [\xi]) = [\mathbf{1}_J(1)\xi_1 : \dots : \mathbf{1}_J(r)\xi_r : 0 : \dots : 0].$$

In particular, this image is independent of the  $z$ -coordinates and  $\Phi^{-1}(\Phi(z, [\xi])) \cong D_J$ .

The restriction  $\Phi|_{p^{-1}(D_I)}$  is injective out of  $\bigcup_{J \subset I} \tilde{D}_J$  if and only if the corresponding map

$$\Phi^\dagger : \mathbb{C}^{n-r} \times (\mathbb{C}^n \setminus \mathbb{C}^r \times \{0\}^{n-r}) \longrightarrow \mathbb{C}^r \times \mathbb{C}^{n-r} \times \mathbb{C}^k$$

is so.

Since the first coordinate map of  $\Phi^\dagger$  is given by the identity  $\mathbb{C}^r \rightarrow \mathbb{C}^r$ , this is equivalent to the injectivity of

$$\begin{aligned} \Phi^\ddagger : \mathbb{C}^{n-r} \times (\mathbb{C}^{n-r} \setminus \{0\}) &\longrightarrow \mathbb{C}^{n-r+k} & (\spadesuit) \\ (z, \xi) &\longmapsto \left( \xi_{r+1}, \dots, \xi_n, \frac{\sum_{j=r+1}^n a_j^1 \xi_j z_j}{a_0^1 + \sum_{j=r+1}^n a_j^1 z_j}, \dots, \frac{\sum_{j=r+1}^n a_j^k \xi_j z_j}{a_0^k + \sum_{j=r+1}^n a_j^k z_j} \right). \end{aligned}$$

We see that  $\Phi|_{p^{-1}(D_I)}$  is injective out of  $\bigcup_{J \subset I} \tilde{D}_J$  whenever the above map  $\Phi^\ddagger : \mathbb{C}^{n-r} \times \mathbb{C}^{n-r} \rightarrow \mathbb{C}^{n-r+k}$  is injective out of the corresponding subset. We recognise in equation  $\spadesuit$  the expression for the morphism associated to the pair  $(D_I, D(I^{\mathbb{C}})|_{D_I})$ . Hence we are reduced to showing that this arrangement of  $(n-r)+k+1$  hyperplanes in  $D_I \cong \mathbb{P}^{n-r}$  has almost ample logarithmic cotangent bundle.

By Lemma 4.3.19 below, we know that this pair already has ample modulo boundary logarithmic cotangent bundle.

The conclusion follows by induction on the dimension, provided that we prove the case  $n = 2$ .

For any  $(z = (0, z_2), [\xi_1 : \xi_2]) \in p^{-1}(D_1 \cap U_0)$ ,  $\blacklozenge$  reads as

$$\Phi(z, [\xi]) = \left[ \xi_1 : \xi_2 : \frac{a_2^1 \xi_2 z_2}{a_0^1 + a_2^1 z_2} : \cdots \right].$$

A quick computation shows that this map is injective in restriction to

$$p^{-1}(D_1 \cap U_0) \setminus \tilde{D}_1 = \{((0, z_2), [\xi_1 : 1])\}. \quad \blacksquare$$

**Lemma 4.3.19**

Let  $D = \sum_{0 \leq i \leq n+k} D_i$  be an arrangement of  $n+k+1$  hyperplanes in general position in  $\mathbb{P}^n$  satisfying Condition  $\star$ . Then so does the arrangement  $D(I^c)|_{D_I}$  in  $D_I \cong \mathbb{P}^{n-|I|}$ , for any subset of indices  $I$  of cardinality  $< n$ .

**Proof:** We assume for simplicity that  $I = \{0, \dots, r\}$  and  $D_i = (Z_i = 0)$  for  $i = 0, \dots, r$ .

First, it is clear that the arrangement  $D(I^c)|_{D_I}$  is in general position (*i.e.* is a normal crossings divisor).

Assume that the hyperplanes  $D_j \cap D_I$ ,  $i = r+1, \dots, n+k$  do not satisfy Condition  $\star$ . Then one can find a symmetric matrix  $Q'$  of size  $n-r$  and rank less than 4 such that

$$A_I^\top Q' A_I = 0,$$

then the matrix

$$Q = \left( \begin{array}{c|c} 0_{r+1} & 0 \\ \hline 0 & Q' \end{array} \right)$$

satisfies

$$A^\top Q A = 0$$

and Condition  $\star$  does not hold for the hyperplanes  $D_i$ 's in  $\mathbb{P}^n$ .  $\blacksquare$

### 4.3.6 Description of the augmented base locus in terms of the number of hyperplanes

The proof of Theorem 4.3.10 actually provides a complete description of the image

$$p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)))$$

of the augmented base locus of the logarithmic cotangent bundle  $\Omega_{\mathbb{P}^n}(\log D)$ .

#### Theorem 4.3.20

The projection

$$p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)))$$

is the union of all ruled quadric surfaces which are simultaneously tangent to all hyperplanes  $H_i$ . Hence, for  $4n - 3 \geq c \geq 3n$  hyperplanes, this locus has dimension  $4n - 1 - c$ , and

$$p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1))) = \mathbb{P}^n$$

if  $c \leq 3n - 1$ .

**Proof:** We know indeed that the locus  $p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)))$  is made of lines. According to the construction of the previous section, the  $(n - 2)$ -plane, dual of any such line, is contained in some quadric scroll in  $\mathbb{P}^{n*}$ , which also contains the hyperplanes  $H_i$ 's as points of the dual projective space. Conversely, for any quadric scroll  $X$  containing the  $H_i$ 's in  $\mathbb{P}^{n*}$ , the dual of any  $\mathbb{P}^{n-2}$  of its rulings is a superjumping line. If  $n \geq 3$ , it is a line of the quadric surface dual to  $X$ .

Consequently, the image of  $\mathbb{B}_+(\mathcal{O}_{\Omega_{\mathbb{P}^n}(\log D)}(1))$  in  $\mathbb{P}^n$  is made of  $\text{Supp } D$  and all the quadric surfaces that are simultaneously tangent to all the components  $H_i$ .

Recall that the dimension of the space of quadrics of rank  $\leq 4$  in  $\mathbb{P}^{n*}$  equals  $4n - 3$ . In the limit case  $c = 4n - 3$ , for a general choice of hyperplanes (here, the term *general* is to be understood as *imposing linearly independent conditions on quadrics*), there exists a unique quadric  $X$  of rank exactly 4 containing all of them in  $\mathbb{P}^{n*}$ . Thus  $\mathbb{B}_+(\mathcal{O}_{\Omega_{\mathbb{P}^n}(\log D)}(1))$  projects onto  $D \cup S$ , where  $S = X^*$  is a smooth quadric surface.

Similarly, the variety of rank  $\leq 4$  quadrics containing  $c$  general points in  $\mathbb{P}^n$  has dimension  $4n - 3 - c$ . We derive that for a general choice of  $c$  hyperplanes,  $p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1))) \setminus \text{Supp } D$  has dimension  $2 + 4n - 3 - c = 4n - 1 - c$ . For  $c < 3n$ , this augmented base locus is not anymore dominant.

We deal separately with the 2-dimensional case, since we do not get surfaces. The 'problematic' lines showing up in the proof are the points of any conic in  $\mathbb{P}^{2*}$  containing all the points  $H_i^*$ . In the original plane  $\mathbb{P}^2$ , if there exists a conic  $\mathcal{C}$  osculating the lines  $H_i$ , then any other tangent line is superjumping. These lines cover exactly  $p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^2}(\log D))}(1)))$ . Indeed, saying that given a point  $q \in \mathbb{P}^2$ , there is a line  $\ell$  tangent to  $\mathcal{C}$  and containing  $q$ , amounts to say that  $q^* \cap \mathcal{C}^* \neq \emptyset$ .

**Remark 4.3.21 (Warning)** The reader should be aware that all the illustrations we give are not completely realistic, for they represent real affine spaces rather than complex projective spaces. In particular, figure 4.4 does not allow to see that the whole plane is actually covered by superjumping lines.

Denote by  $\Sigma$  the closure

$$\overline{p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}^n(\log D)}(1))) \setminus \text{Supp } D}$$

of the complement of  $\text{Supp } D$  in the projection of the augmented base locus of the logarithmic cotangent bundle. We have seen that  $\Sigma$  consists of a union of lines. In addition, the superjumping lines were identified, through the construction of the previous section, as the lines of all quadric surfaces  $S$  contained in some  $\mathbb{P}^3 \subset \mathbb{P}^n$ , which are simultaneously tangent to all components of  $D$ .

It is known that any irreducible quadric surface is ruled (it is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$ ). Any surface  $S$  as above is the union of its lines, and is thus entirely contained in  $\Sigma$ . We have ultimately described the locus  $\Sigma$  as a union of quadric surfaces.

To be more precise, we will determine the dimension of the family of such surfaces  $S$ , depending on the number of components of  $D$ . Recall from the previous sections that such a surface  $S$  is the dual variety of a quadric hypersurface in  $\mathbb{P}^{n*}$  of rank  $\leq 4$  containing all the points  $p_i = H_i^*$ .

First, let us compute the dimension of the variety  $\Xi$  of quadric hypersurfaces of rank  $\leq 4$  in  $\mathbb{P}^n$ . To do so, we use the characterisation of quadrics of rank at most 4 as quadric hypersurfaces containing an  $(n-2)$ -plane. The variety  $\Xi$  can then be represented as the image under the second projection of the incidence variety

$$\mathcal{I} = \{(Z, X); Z \subset X\} \subset \mathbb{G}(n-2, n) \times \mathbb{P}^{\binom{n+2}{2}-1}.$$

Fix an  $(n-2)$ -plane  $Z \subset \mathbb{P}^n$ . What can be the dimension of the fibre  $\pi_1^{-1}(Z)$  of the first projection above  $Z$ ? Observe that this fibre is actually the kernel of the linear map

$$\mathbb{C}^{\binom{n+2}{2}} \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2)) \rightarrow H^0(Z, \mathcal{O}_Z(2)) \cong \mathbb{C}^{\binom{n}{2}}.$$

Hence  $\pi_1^{-1}(Z)$  has dimension  $\binom{n+2}{2} - \binom{n}{2} - 1 = 2n$ .

In the end, we find  $\dim \mathcal{I} = 2n + \dim \mathbb{G}(n-2, n) = 2n + 2(n-1) = 4n-2$ . Since the fibre above a general point of  $\Xi$  has dimension 1, we compute  $\dim \Xi = 4n-3$ .

We see now that for a general choice of  $c \geq 4n-2$  hyperplanes in  $\mathbb{P}^n$ , the image  $p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}^n(\log D)}(1)))$  of the augmented base locus is empty, since  $4n-2$  general points can not belong to a same quadric of rank  $\leq 4$ . If  $D$  is made of  $c \leq 4n-3$  general components, then the locus  $\Sigma$  is swept out by a  $(4n-3-c)$ -dimensional family of quadric surfaces, and thus has dimension  $4n-3-c+2 = 4n-1-c$  for  $c \geq 3n$ . When we have strictly less than  $3n$  components,  $\Sigma$  is the whole space  $\mathbb{P}^n$ . ■

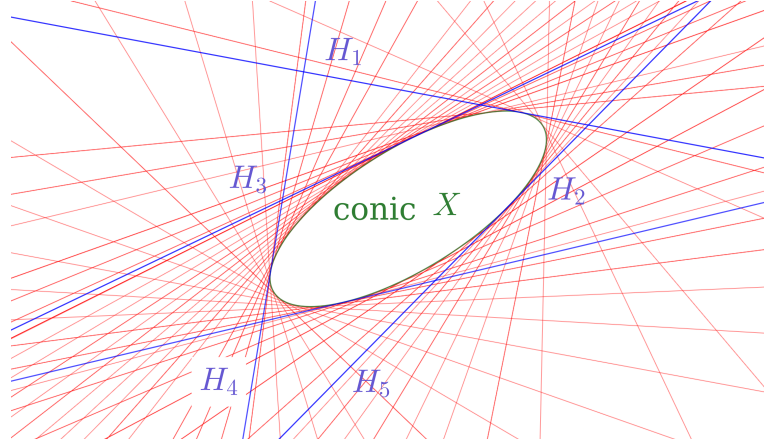


Figure 4.4: Lines in  $\mathbb{P}^2$  tangent to a same conic:  $p(\mathbb{B}_+(\mathcal{O}_{X_1}(1))) = \mathbb{P}^2$  is covered by the red lines (this is not clear on the picture since it represents  $\mathbb{R}^2$ )

When  $c \leq 3n - 1$ ,  $\mathbb{P}^n$  is entirely covered by superjumping lines. However, the augmented base locus  $\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1))$  upstairs can still be a proper subset of  $\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))$ . Is it possible to describe it more precisely?

Assume that  $2n \leq c \leq 3n - 1$ . The image  $p(\mathbb{B}_+(\mathcal{O}(1)))$  is swept out by a  $(4n - 3 - c)$ -dimensional family of  $\mathbb{P}^1 \times \mathbb{P}^1$ 's tangent to all the hyperplanes. Therefore, the set of superjumping lines is a subvariety of  $\mathbb{G}(1, n) \cong \mathbb{P}^{2n-2}$  of dimension  $4n - 2 - c$ . We remark that for  $c \leq 2n$ , all lines in  $\mathbb{P}^n$  are superjumping.

#### Theorem 4.3.22

Assume that we have  $2n + 1 \leq c \leq 3n - 1$  hyperplanes  $H_i$  in general position in  $\mathbb{P}^n$ . Then the augmented base locus

$$\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)) \subset \mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))$$

has dimension  $\leq 4n - 1 - c$ .

**Proof:** We first show that  $\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1))$  equals exactly the union over all superjumping lines  $l$  of the curves  $\mathbb{P}(\mathcal{O}_l)$  induced by trivial quotients of  $\Omega_{\mathbb{P}^n}(\log D)$ .

Write  $c = n + k + 1$  with  $k \leq 1$  and let the notation be as in subsection 4.3.3. We work in the standard affine coordinates associated to the open subset  $\mathbb{P}^n \setminus H_0 = \{Z_0 \neq 0\}$ .

Let

$$\left(x, \xi = \sum_{i=1}^n \xi_i \frac{\partial}{\partial z_i}\right) \in \mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)) \setminus p^{-1}(D)$$

and denote for all  $i = 1, \dots, n$  and  $j = 1, \dots, k$ ,

$$V_i = \xi_i, V_j = \frac{\sum_{i=1}^n a_i^j x_i \xi_i}{\sum_{i=0}^n a_i^j x_i},$$

so that  $\Phi(x, \xi) = [V_1 : \dots : V_{n+k}]$ . Then the fibre  $Z = \Phi^{-1}(\Phi(x, \xi))$  is positive-dimensional and  $x$  belongs to some superjumping line  $l \subset p(Z)$ . The inclusion

$l \subset p(Z)$  shows that any  $z = (z_1, \dots, z_n) \in l \setminus H_0$  satisfies the equations

$$\sum_{i=0}^n a_i^j z_i (V_{n+j} - V_i) = 0.$$

One can extend the vector field

$$\xi(z) = \sum_{i=1}^n \frac{\xi_i}{x_i} z_i \frac{\partial}{\partial z_i}$$

defined on  $l \setminus H_0$  as a never-vanishing holomorphic global section of  $T_{\mathbb{P}^n}(-\log D)|_l$ . In other words, there is a trivial quotient  $\mathcal{O}_l$  of  $\Omega_{\mathbb{P}^n}(\log D)|_l$  such that  $(x, \xi) \in \mathbb{P}(\mathcal{O}_l)$ .

In order to compute the dimension of  $\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1))$ , we will ‘count’ all such curves  $\mathbb{P}(\mathcal{O}_l)$ .

Above each surface  $\mathbb{P}^1 \times \mathbb{P}^1$  composing  $p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)))$  lives a 1-parameter family of  $\mathbb{P}(\mathcal{O}_l)$  inside  $\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1))$ . Therefore, the augmented base locus has dimension at most  $4n - 2 - c + 1 = 4n - 1 - c$ .  $\blacksquare$

If  $c \leq 2n$ , the morphism  $\Phi : \mathbb{P}(\Omega_{\mathbb{P}^n}(\log D)) \rightarrow \mathbb{P}^{c-2}$  cannot be generically finite, since  $c - 2 \leq 2n - 2 < 2n - 1 = \dim \mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))$ . We deduce

**Corollary 4.3.23** *The logarithmic cotangent bundle associated to  $c$  general hyperplanes is big if and only if  $c \geq 2n + 1$ .*

**Remark 4.3.24** *Note that  $c = 2n + 1$  is also the bound for Brody-hyperbolicity of the complement  $\mathbb{P}^n \setminus D$ .*

## 4.4 The orbifold case

In this section, we study an orbifold version of the previous setting.

We consider an orbifold pair

$$\left( \mathbb{P}^n, \Delta = \sum_{i=0}^{n+k} (1 - 1/m_i) H_i \right)$$

where  $H_0, \dots, H_{n+k}$  is an arrangement of hyperplanes in general position. The associated orbifold cotangent bundle  $\Omega_{(\mathbb{P}^n, \Delta)}$  was defined in section 3.2.

Recall that the augmented base locus of  $\Omega_{(X, \Delta)}$  is defined by

$$\mathbb{B}_+(\Omega_{(X, \Delta)}) = \bigcap_{N \geq 1} \bigcap_{p/q \in \mathbb{Q}} \text{Bs}(S^{[Nq]} \Omega_{(X, \Delta)} \otimes A^{-Np})$$

for some ample line bundle  $A$  over  $X$ .

### Definition 4.4.1

*The orbifold cotangent bundle  $\Omega_{(\mathbb{P}^n, \Delta)}$  is said ample modulo  $|\Delta|$  if  $\mathbb{B}_+(\Omega_{(X, \Delta)}) \subset |\Delta|$ .*

As in the logarithmic case, it is important to note that the orbifold cotangent bundle cannot be ample when  $\text{Supp } \Delta \neq \emptyset$ .

**Lemma 4.4.2** (see [31], Lemma 3.7) *Let  $(\mathbb{P}^n, \Delta)$  be a smooth orbifold pair. Then for any strictly adapted covering  $\pi : Y \rightarrow \mathbb{P}^n$ , the orbifold cotangent bundle  $\Omega_{(\mathbb{P}^n, \Delta)}$  has negative quotients supported on each component of  $\Delta$  with finite multiplicity and trivial quotients supported on each component with infinite multiplicity.*

Thus ampleness modulo boundary is the strongest property we can hope for.

#### 4.4.1 The Fermat cover for hyperplane arrangements

Assume that the hyperplanes have equal orbifold multiplicities. In this particular case, we can construct a global strictly adapted covering as follows.

For any  $i = 0, \dots, n+k$ , let  $\ell_i$  be a linear form defining  $H_i$ . Without loss of generality, we may assume that  $\ell_i(Z) = Z_i$  for  $i_0, \dots, n$ , so that  $H_0, \dots, H_n$  are the coordinates hyperplanes, and we write

$$\ell_{n+j}(Z) = \sum_{i=0}^n \alpha_i^j Z_i, j = 1, \dots, k.$$

In the projective space  $\mathbb{P}^{n+k}$ , one can identify  $\mathbb{P}^n$  with the linear subspace

$$\bigcap_{j=1}^k \{X_{n+j} = \sum_{i=1}^n a_{i,j} X_i\}.$$

Let  $Y$  be the complete Fermat intersection

$$\bigcap_{j=1}^k \{X_{n+j}^m = \sum_{i=1}^n a_{i,j} X_i^m\}.$$

This is an  $n$ -dimensional subvariety of  $\mathbb{P}^{n+k}$ . Then the holomorphic map

$$\pi: \begin{array}{ccc} Y & \longrightarrow & \mathbb{P}^n \\ [X_0 : \dots : X_{n+k}] & \longmapsto & [X_0^m : \dots : X_n^m] \end{array}$$

realises  $Y$  as a strictly adapted cover of the pair  $(\mathbb{P}^n, \Delta)$ , that is, a Galois covering ramifying exactly over the hyperplanes  $H_i$  with ramification order  $m$ . This is the *Fermat cover* associated with the pair  $(\mathbb{P}^n, \Delta)$ .

### 4.4.2 Ampleness of the orbifold cotangent bundle

In this context, we will prove the following analogue of 4.3.12.

#### Theorem 4.4.3

If the arrangement satisfies  $\star$  and for all  $i$ ,  $\bar{m}_i \geq 2n$ , then the orbifold cotangent bundle  $\Omega_{(\mathbb{P}^n, \Delta)}$  is ample modulo  $|\Delta|$ .

**Proof:** To begin with, increasing the multiplicities  $m_i$  will always increase the ampleness of  $\Omega(\mathbb{P}^n, \Delta)$  (see [31]). Indeed, consider two orbifolds divisors with the same support

$$\Delta = \sum_{i=0}^{n+k} (1 - 1/m_i) H_i, \Delta' = \sum_{i=0}^{n+k} (1 - 1/m'_i) H_i$$

such that for each  $i$ , one has  $m_i \leq m'_i$ . We can find a ramified covering  $\pi : X \rightarrow \mathbb{P}^n$  which is adapted for both  $\Delta$  and  $\Delta'$ . Namely, we have for each  $i$ ,

$$\pi^* D_i = p_i m_i E_i = p'_i m'_i E_i.$$

Thanks to the local expression of the sections given in (3.1), we see that  $\Omega(\pi, \Delta)$  is a subsheaf of  $\Omega(\pi, \Delta')$ , and similarly  $S^{[q]} \Omega_{(\mathbb{P}^n, \Delta)}$  is a subsheaf of  $S^{[q]} \Omega_{(\mathbb{P}^n, \Delta')}$ . This implies

$$\mathbb{B}_+(\Omega_{(X, \Delta')}) \subset \mathbb{B}_+(\Omega_{(X, \Delta)}).$$

Therefore, we can assume without loss of generality that all the  $m_i$  are equal.

Recall that in subsection 4.3.3, we characterised a point  $[V_1 : \dots : V_{n+k}] \in \mathbb{P}^{n+k-1}$  in the image  $W$  of the map  $\Phi$  associated with  $\Omega_{\mathbb{P}^n}(\log |\Delta|)$  by the following equations, for some  $(z, [\xi]) \in \mathbb{P}(\Omega_{\mathbb{P}^n}(\log |\Delta|))$ . We work here in the open subset  $\{Z_0 \neq 0\}$  with standard affine coordinates.

$$\begin{aligned} \left\{ \begin{array}{l} V_1 = \xi_1 \\ \vdots \\ V_n = \xi_n \\ (a_0^j + \sum_{i=1}^n a_i^j z_i) V_{n+j} = \sum_{i=1}^n a_i^j z_i \xi_i \end{array} \right. & \iff \left\{ \begin{array}{l} V_1 = \xi_1 \\ \vdots \\ V_n = \xi_n \\ a_{0,j} V_{n+j} + \sum_{i=1}^n a_i^j z_i (V_{n+j} - V_i) = 0 \end{array} \right. \\ & \iff \ker M \neq \{0\} \\ & \iff \operatorname{rk}(M) < n + 1. \end{aligned}$$

Here  $M$  is the  $(n+1) \times k$ -matrix

$$\begin{pmatrix} a_0^{n+1}(V_{n+1} - V_0) & \cdots & a_n^{n+1}(V_{n+1} - V_n) \\ \vdots & \ddots & \vdots \\ a_0^{n+k}(V_{n+k} - V_0) & \cdots & a_{n,n+k}(V_{n+k} - V_n) \end{pmatrix},$$



setting  $z_0 = 1, \xi_0 = V_0 = 0$ . Hence  $W$  is described by the vanishing of the maximal minors of  $M$ . Moreover, these computations characterise the positive dimensional fibres of  $\Phi$  as the points  $V$  where

$$\text{rk}(M) < n.$$

Denote by  $p$  the projection  $\mathbb{P}(\Omega_{\mathbb{P}^n}(\log|\Delta|)) \rightarrow \mathbb{P}^n$ . Our goal is to exhibit global orbifold forms vanishing on an ample divisor, whose base locus is contained in

$$p\left(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1))\right).$$

Consider a component of  $D$ , for instance  $H_1 = (z_1 = 0)$ .

The image  $\Phi(p^{-1}(H_1))$  is described by the equations

$$\begin{aligned} \left\{ \begin{array}{l} V_1 = \xi_1 \\ \vdots \\ V_n = \xi_n \\ (a_0^j + \sum_{i=2}^n a_i^j z_i) V_{n+j} = \sum_{i=2}^n a_i^j z_i \xi_i \end{array} \right. & \iff \left\{ \begin{array}{l} V_1 = \xi_1 \\ \vdots \\ V_n = \xi_n \\ a_0^j V_{n+j} + \sum_{i=2}^n a_i^j z_i (V_{n+j} - V_i) = 0 \end{array} \right. \\ & \iff \ker M' \neq \{0\} \\ & \iff \text{rk}(M') < n \end{aligned}$$

where  $M'$  is the  $n \times k$ -matrix

$$\begin{pmatrix} a_0^1(V_{n+1} - V_0) & a_2^1(V_{n+1} - V_2) & \cdots & a_n^1(V_{n+1} - V_n) \\ \vdots & \ddots & \vdots & \\ a_0^k(V_{n+k} - V_0) & a_2^k(V_{n+k} - V_2) & \cdots & a_n^k(V_{n+k} - V_n) \end{pmatrix}.$$

Again, a point  $V \in \mathbb{P}^{n+k-1}$  belongs to  $\Phi(\pi^{-1}(H_1))$  if all the  $n \times n$ -minors vanish at  $V$ . Hence, any  $n \times n$ -minor  $\Pi$  satisfies  $\Phi^*\Pi|_{\pi^{-1}(H_1)} \equiv 0$ , so that

$$(p^*\ell_1)^{-1}\Phi^*\Pi \in H^0\left(\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D)), \mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(n) \otimes p^*\mathcal{O}_{\mathbb{P}^n}(-H_1)\right).$$

We deduce that for each such  $\Pi$

$$\text{Bs}(\Phi^*\Pi) \subset p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1))).$$

For instance, let  $\Pi$  be the minor of  $M'$  made of the first  $n$  rows.

Then  $(p^*\ell_1)^{-1}\Phi^*\Pi$  corresponds to a global symmetric form

$$\omega \in H^0(\mathbb{P}^n, S^n\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(-1)).$$

The form  $\omega$  has poles only along the hyperplanes  $H_0, H_2, \dots, H_{2n}$ . Moreover, each monomial in  $\omega$  has at most a simple pole along each component  $H_i$ .

Denote

$$\eta = \ell_0 \ell_2 \cdots \ell_{2n} \omega^{2n} \in H^0(\mathbb{P}^n, S^{2n^2} \Omega_{\mathbb{P}^n}(\log D)).$$

We will show that  $\eta$  is an orbifold form as soon as  $m \geq 2n$ .

Indeed, recall the Fermat cover  $\pi : Y \subset \mathbb{P}^{n+k} \rightarrow (\mathbb{P}^n, \Delta)$  given by

$$\pi([X_0 : \cdots : X_{n+k}]) = [X_0^m : \cdots : X_n^m].$$

In these coordinates, for all  $i$ , we have  $\pi^* \ell_i = X_i^m$  and  $\pi^* \frac{d\ell_i}{\ell_i} = m \frac{dX_i}{X_i}$ . Hence

$$\pi^* \eta = X_0^m X_2^m \cdots X_{2n}^m (\pi^* \omega)^{2n}.$$

But we know that each monomial in  $\omega^{2n}$  has pole order at most  $2n$  along each component  $H_i$ , so that we can write

$$\pi^* \eta = (X_0 X_2 \cdots X_{2n})^{m-2n} \tilde{\omega} \in H^0(Y, S^{2n^2} \Omega_Y)$$

for some holomorphic  $\tilde{\omega}$ . Hence, we have constructed an orbifold form vanishing along an ample divisor as

$$\omega^{2n} = (\ell_0 \ell_2 \cdots \ell_{2n})^{-1} \eta \in H^0(\mathbb{P}^n, S^{[2n^2]} \Omega_{(\mathbb{P}^n, \Delta)} \otimes \mathcal{O}_{\mathbb{P}^n}(-2n)).$$

As a consequence of this computation, we get the inclusion

$$\mathbb{B}_+(\Omega_{(\mathbb{P}^n, \Delta)}) \subset \text{Bs}(\omega^{2n}) \subset p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}^n(\log D)}(1))).$$

We conclude that for multiplicity  $m \geq 2n$ , the orbifold cotangent bundle of the pair  $(\mathbb{P}^n, \Delta)$  is ample modulo boundary whenever  $\Omega_{\mathbb{P}^n}(\log |\Delta|)$  is, and the statement of the theorem follows from [4.3.12](#). ■

Note the difference between our orbifold forms and the ones of [\[31\]](#). While the latter are explicitly constructed in coordinates, ours have a geometric interpretation and directly come from some specific logarithmic forms.

## 4.5 Applications to complex hyperbolicity

In this section, we give some applications of our results to complex hyperbolicity. Note that we do not claim any originality in part of the proofs below, which are exactly the same that in [\[31\]](#).

### 4.5.1 Orbifold hyperbolicity

We have proved the inclusion

$$\mathbb{B}_+(\Omega_{(\mathbb{P}^n, \Delta)}) \subseteq p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}^n(\log D)}(1))).$$

From the discussion in section [4.3.6](#), we deduce a bound on the dimension of the orbifold augmented base locus  $\mathbb{B}_+(\Omega_{(\mathbb{P}^n, \Delta)})$  away from  $\text{Supp } \Delta$ , in case  $\Omega_{(\mathbb{P}^n, \Delta)}$  is not ample modulo  $\text{Supp } \Delta$ .

As in [\[31\]](#), we infer from our Theorem [4.4.3](#) a result about algebraic degeneracy of orbifold curves.

**Corollary 4.5.1**

Consider the orbifold pair

$$\left( \mathbb{P}^n, \Delta = \sum_{1 \leq i \leq c} \left( 1 - \frac{1}{m_i} \right) H_i \right)$$

formed by an arrangement of  $c$  hyperplanes in general position with orbifold multiplicities  $m_i \geq 2n$ . If the arrangement satisfies  $\star$ , then the orbifold pair  $(\mathbb{P}^n, \Delta)$  is Brody-hyperbolic. In fact,  $(\mathbb{P}^n, \Delta)$  is even Kobayashi-hyperbolic.

This result is not new and follows from Nochka's Theorem below. Since our proof completely differs, though, we have chosen to include it.

**Theorem 4.5.2 (Nochka [83])**

Let  $f : \mathbb{C} \rightarrow \mathbb{P}^n$  be a holomorphic map and let  $d$  be the minimal integer such that the image of  $f$  is contained in a  $d$ -dimensional subspace. Let  $H_1, \dots, H_q$  be hyperplanes in general position in  $\mathbb{P}^n$ . Assume that the curve  $f$  intersects each  $H_i$  with multiplicity  $m_i$ . Then

$$\sum_{i=1}^q \left( 1 - \frac{d}{m_i} \right) < 2n - d + 1.$$

As a consequence, if  $q \geq 2n+2$  and  $m_i \geq 2n$ , the orbifold pair  $(\mathbb{P}^n, \Delta = \sum_{i=1}^q (1 - 1/m_i) H_i)$  is Brody-hyperbolic.

**Proof:** According to Theorem 3.3.9, if  $(\mathbb{P}^n, \Delta)$  is not hyperbolic, there exists either a non-constant orbifold curve  $f : \mathbb{C} \rightarrow (\mathbb{P}^n, \Delta)$  or an orbifold curve inside a stratum  $(\Delta_I, \Delta(I^{\mathbb{C}})|_{\Delta_I})$ . By 4.4.3 and 3.2.7, all orbifold entire curves  $\mathbb{C} \rightarrow (\mathbb{P}^n, \Delta)$  are constant. Hence we are left with the second possibility. But according to Lemma 4.3.19,  $(\Delta_I, \Delta(I^{\mathbb{C}})|_{\Delta_I})$  is again an orbifold pair satisfying the conditions of 4.4.3, so that it cannot contain any non-constant orbifold curve.  $\blacksquare$

We use the description of the augmented base locus provided in 4.3.6 to refine 4.5.1.

**Theorem 4.5.3**

With the same assumptions, if the hyperplanes impose at least  $3n + k$  independent conditions on quadrics, with  $1 \leq k \leq n - 3$ , then there exists a Zariski-closed subset  $\Sigma$  of dimension  $k + 2$  containing all orbifold curves  $f : \mathbb{C} \rightarrow (\mathbb{P}^n, \Delta)$ .

**4.5.2 Hyperbolicity of Fermat covers**

Fermat hypersurfaces are one class of varieties for which several hyperbolicity results have been obtained. For instance, one has the two results of Green.

**Theorem 4.5.4 ([70], Ex. 3.10.21)**

Let

$$F(n, d) = \left\{ [Z_0 : \dots : Z_{n+1}]; Z_0^d + \dots + Z_{n+1}^d = 0 \right\} \subset \mathbb{P}^{n+1}$$

be the Fermat hypersurface of degree  $d$  in  $\mathbb{P}^{n+1}$ .

1. If  $d \geq (n+1)^2$ , then every entire curve  $f : \mathbb{C} \rightarrow F(n, d)$  has its image contained in a linear subspace of dimension  $\lfloor n/2 \rfloor$ .
2. If  $d > (n+1)(n+2)$ , then every entire curve  $f : \mathbb{C} \rightarrow \mathbb{P}^{n+1} \setminus F(n, d)$  has its image contained in a linear subspace of dimension  $\lfloor (n+1)/2 \rfloor$ .

These results are consequences of *Cartan's truncated defect relation* (see [70], 3.B.42) which, with the orbifold terminology, gives the linear degeneracy of orbifold curves inside an orbifold pair  $(\mathbb{P}^n, \sum_{0 \leq i \leq n+1} (1 - 1/m)H_i)$  attached to an arrangement of  $n+2$  general hyperplanes, provided that  $\sum_i (1 - n/m)^+ > n+1$ . The hypersurface  $F(n, m)$  is precisely the Fermat cover associated with this orbifold.

We use now 4.4.3 to prove the hyperbolicity of Fermat covers with different assumptions.

**Theorem 4.5.5**

The Fermat cover associated with an arrangement of hyperplanes in  $\mathbb{P}^n$  imposing at least  $4n - 2$  linearly independent conditions on quadrics, with ramification  $m \geq 2n$ , is Kobayashi-hyperbolic.

**Proof:** Let  $\pi : Y \rightarrow (\mathbb{P}^n, \Delta)$  be the Fermat cover. It suffices to prove the Brody-hyperbolicity.

Let  $f : \mathbb{C} \rightarrow Y$  be an entire curve. According to 4.4.3, its image  $f(\mathbb{C})$  lies in the ramification locus of the covering  $\pi$ .

Note that the ramification locus can be seen as the Fermat cover associated to an arrangement of  $d \geq 4n - 3$  hyperplanes in  $\mathbb{P}^{n-1}$ . By Lemma 4.3.19 above, one can still assume that the assumptions of 4.4.3 are satisfied. Hence we obtain the hyperbolicity of  $Y$ . ■

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